THE CONTINUUM HYPOTHESIS AND THE SET-THEORETIC MULTIVERSE

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> *For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality[...].*
> –Kurt Gödel (1947), *What is Cantor's Continuum Problem?*

> *It is insufficient to present a beautiful landscape, a shining city on a hill, for we are widely traveled and know that it is not the only one.*
> –Joel Hamkins (2015), *Is the Dream Solution of the Continuum Hypothesis Attainable?*

1. Introduction

Recently, some set theorists and philosophers of mathematics have begun championing a new understanding of set theory. The question turns in part on whether there is only one correct or legitimate theory of sets or not. The more traditional *monists* think that the answer (ultimately) is 'yes'. *Pluralists* argue 'no,' and think that we have reason to accept several set theories.

In this essay I want to achieve a better understanding of the monist and pluralist understanding of set theory. I start by reviewing one out of the claims that have been shown independent of the Zermelo-Fraenkel with Choice axiomatization of set theory (ZFC), namely Cantor's Continuum Hypothesis (CH). This claim has both inspired monists to search for new axioms so as to, paraphrasing Gödel, achieve a more complete description of set theoretic reality, and also pluralists in exploring the notion that the claim might somehow have an indeterminate truth value. After having stated CH, I go on to discuss Gödel's and Maddy's monist position regarding set theory and CH, and contrast their views with Hamkins' (2012, 2015) pluralism about set theory. All of these theorists are arguably realists (although Maddy (1997) is less concerned with questions of ontology as opposed to Maddy (1990)). Since part of the goal of the essay is to get clearer on what set theoretic monism and pluralism might be more generally, I end by pointing towards other possible monist and pluralist positions less committed to a full-blooded realist picture of what set theory is about.

2. The Continuum Hypothesis

We start by looking at CH, where the following presentation draws mainly on Enderton (1977: Chapter 6) but also Gödel (1947/1964).

The cardinal number κ of a set A denotes the size of that set. Two sets A and B are said to have the same size if they are *equinumerous* (written $A \approx B$), which means that there is a one-to-one function from A onto B. So we can say that card A = card B iff $A \approx B$.[1] For any finite-sized set, the cardinal number of that set will be one of the natural numbers. But the set of all natural numbers itself, $\omega = \{0, 1, 2, 3,...\}$, is an infinite set. What, then, should we say about the cardinal number of ω? One of the strengths of set theory, as it originated in the work of Cantor, is that it allows us to speak of the sizes of infinite sets.

The cardinal number of ω is called $\aleph_0$, which is the least infinite cardinal. In fact, many infinite-sized sets have cardinality $\aleph_0$, like the set of all odd numbers, the set of all integers, the set of all rational numbers, and many more. But there are also infinite sets with cardinalities strictly greater than $\aleph_0$. One such set is the set of real numbers $\mathbb{R}$. It can be shown that $\mathbb{R} \approx \mathscr{P}\omega$, where $\mathscr{P}\omega$ is the powerset of the set of natural numbers.[2] By looking at Cantor's theorem (6B(b) in Enderton 1977), which states that for any given set A, the powerset of A is strictly greater than A, that is, $\forall A[\text{card } A < \text{card } \mathscr{P}A]$, we know

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that $\aleph_0 <$ card $\mathbb{R}$. Since it also holds that card $\mathbb{R} = 2^{\aleph_0}$, we can say $\aleph_0 < 2^{\aleph_0}$.

To sum up so far, we have that:

i) card $\omega = \aleph_0$

ii) card $\mathbb{R} = 2^{\aleph_0}$

iii) $\aleph_0 < 2^{\aleph_0}$

We are now ready to state CH. Here is a rather simple question to ask when faced with (iii): is there any cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$? Cantor hypothesized that the answer is negative; there is no such . If we define $\aleph_1$ to be the least infinite cardinal such that $\aleph_0 < \aleph_1$, CH can be stated as

$$2^{\aleph_0} = \aleph_1.$$

The *generalized* CH states that for any infinite cardinal κ there is no cardinal number λ such that $\kappa < \lambda < 2^{\kappa}$.[3] This statement entails CH.

Having hypothesized the claim, it was Cantor's hope to prove CH but he was never able to do so. During the twentieth century it became clear that CH can't be proven from the most popular standard axiomatization of set theory, namely ZFC. The reason for this is that CH (together with several other set theoretical claims) is independent of ZFC. In 1939 Gödel showed that, assuming ZFC is consistent, ZFC+CH is consistent. Thus we cannot disprove CH from ZFC. In 1963 Paul Cohen showed that, assuming ZFC is consistent, ZFC+¬CH is consistent. Thus we cannot prove CH either. Together these results establish that, assuming ZFC is consistent, CH can neither be proven nor disproven from ZFC, and is therefore independent of the axiomatic system.[4] An interesting question to ask, both for the mathematician and the philosopher, is: what to do in light of this fact?

This question is easily related to issues of truth and meaning in mathematics. For example, if one thinks that ZFC exhausts our conception of sets, then it is not clear what to do in light of the independence of CH. Maybe we cannot decide the question. In a similar vein one might wonder if not asking questions about the truth-value of mathematical claims, outside of a given axiomatic system, involves commitment to an unwarranted realism about mathematics. Such a doubt is maybe expressed by Enderton when he writes:

> Indeed, one might well question whether there is any meaningful sense in which one can say that the continuum hypothesis is either true *or* false for the "real" sets. (Enderton 1977:166)

Despite this, some set theorists (like Gödel 1947/1964) do think that CH makes sense, and ought to have a truth-value. This again relates to mathematical methodology: If we think that CH is meaningful and either true or false, then, although ZFC cannot decide the question as to which truth-value, maybe some stronger axiomatization ZFC+X can (where X is any proposed axiom candidate). So, we should look for further axiom candidates that can be added to ZFC. But say you are faced with at least two competing extensions of ZFC, ZFC+X and ZFC+Y, such that ZFC+X entails CH and ZFC+Y entails ¬CH, is it so that at most one of them is true?

In the rest of the essay I turn to an exploration of two competing views of truth in mathematics, with CH as the contested claim. I start with the *monism* of Gödel and Maddy.

3. Monism

Gödel (1947/1964) contemplates the situation regarding CH and claims that even if one could show that CH is independent of ZFC (which it is), this is not a satisfying solution. Gödel thinks instead that this independence would only show us that ZFC is an incomplete theory as it stands. His hope is that new axioms, in line with our conception of sets, might settle the question.

Why does Gödel think this? In part the answer turns on Gödel's realism about the objects of set theory, and mathematics more generally. If one thinks there is an objectively existing realm of sets, only partly described by ZFC, then independent questions like CH, if they have the right semantic relationship with this reality, ought to have a truth-value.[5] Usually, we think that a declarative and meaningful sentence about something objective, like 'earth is the third planet from the sun' or '2 + 2 = 4,' is either true or false but not both. That is, we think there is a unique truth-value for such claims. Gödel thinks similarly for CH.

One reason for this is that CH is about the size of two very important and familiar mathematical objects, as it is about the size of the set of all natural numbers and the size of the set of all real numbers. As Gödel points out, CH can be stated (paraphrasing a bit): Any infinite subset of the continuum can be one-to-one correlated with either the set of all natural numbers or the whole of the continuum. If any mathematical objects (and their properties) have objective existence at all, these sets of numbers are reasonable candidates. Furthermore, if one thinks that this claim of transfinite number theory seems meaningful and either true or false, then looking for an answer in set theory seems to be the best shot. Of course, ZFC can't do the job, but (as stated earlier) maybe some supplemented axiomatic system ZFC+X can.

Say you are a realist, then, how do you justify extending ZFC? We cannot add any old axiom candidate that comes along. This is important to the monist because the addition of further axioms are supposed to yield a better theory of sets and a more complete description of set theoretical reality, so, presumably, we can only add axioms we are justified to hold as true. In Gödel (1947/1964) we find two suggested ways of approaching the question about justification, more fully developed by Maddy (1990, 1997). The first approach is by appeal to *intrinsic* evidence, the second approach is by appeal to *extrinsic* evidence.

Intrinsic evidence: We have intrinsic evidence in set theory when our intuition about sets or our analysis of the concept of set supports a given axiom candidate. Intuitive and conceptual evidence is unified by a kind of directness, that is, there is some immediate grasp of the grounds for justification of the given claims. It was Gödel's belief that we have a kind of perception-like awareness or intuition of the objects of set theory (see especially the supplement to Gödel 1964). This intuition enables us to grasp some of the elementary truths of mathematics, and gives us the data we need to theorize systematically. An axiom candidate X has support from intrinsic evidence when X seems true to us just by contemplating in general our intuitive notion of a set. Of the axioms already part of ZFC, many of them arguably have this kind of intrinsic appeal. As an example, we can think of the axiom of pairing, which states that for any two sets *A* and *B*, there exists the pair set {*A*, *B*}. This seems perfectly in line with our notion of set-formation as the procedure of collecting objects, treating the collection itself as a further object.

Is there any new and intrinsically plausible axiom candidates? The question is difficult to answer: According to Penelope Maddy, there is now almost complete consensus among set theorists that some version of a so-called large cardinal axiom should be added to ZFC. Gödel argued that such an axiom has intrinsic appeal in virtue of our conception of the hierarchy of sets, captured in part by ZFC. Unfortunately, large cardinal axioms do not decide CH. So, if we ask for an intrinsically plausible axiom candidate that decides CH, the answer after decades of searching seems to be 'no.' Although many further axiom candidates have been proposed and explored (Maddy 1997:73–81), none of them have, as the intrinsically evident ones are supposed to do, 'force[d] themselves upon us as being true' (Gödel 1964:484).

Extrinsic evidence: In light of this, modern-day monists (like Maddy) have shifted their attention to the second suggestion made by Gödel, namely that new axiom candidates can be extrinsically justified via their fruitfulness in mathematics (and maybe beyond). For example, if an axiom candidate X greatly simplifies our theory, better explains the intrinsic data or engenders verifiable consequences throughout mathematics, this lends support to X. Adding to this, it seems most axiom candidates can be supported by both intrinsic and extrinsic evidence to different degrees, with some being more intrinsically justified and others more extrinsically justified. Of the established axioms of ZFC, the axiom of choice and the axiom of regularity seem to be the ones with the least intuitive evidence, yet one can argue for both by appealing to extrinsic evidence.[6] The axiom of choice has been shown to be essential in the proof of many important theorems inside and outside of set theory, and, arguably, the axiom of regularity makes for a simpler and more elegant picture of the hierarchy of sets (at least within pure set theory).

When it comes to proposing new axiom candidates, appeal to extrinsic justification is used but has not as of yet established any single extension of ZFC as the definitive one. Let me note here that these kinds of arguments in axiomatic set theory seem to be less powerful than arguments from intrinsic plausibility, at least so far. Unless the fruitfulness or usefulness of a relevant axiom candidate becomes so great that it is worse to do without, an axiom candidate usually needs to be backed by an increased sense that there might be something intuitive about the claim after all if it is to be accepted by most mathematicians.[7]

In sum, then, the monist hopes that there is some axiom candidate X with either great amounts of intrinsic plausibility or extrinsic evidence in its favour (or the right combination of both) so as to outdo its competitors. If ZFC+X entails either CH or ¬CH, the monist has attained what Hamkins (2015) calls the dream solution of the continuum hypothesis. According to the pluralist the dream solution is just that, a dream.

4. Pluralism

We said earlier that Gödel's optimism in the quest for new axioms was in part due to his realist views, but as we will see, realism alone, arguably, does not imply monism.

The underlying conception of ZFC most often appealed to is the so-called iterative conception of sets. Here one starts with the empty set and iterate, along the ordinals, the power-set operation at successor stages and the union operation at limit stages. This gives rise to the cumulative hierarchy *V*. Although most of the axioms of ZFC are in line with this picture, they, as we saw earlier, do not specify all relevant properties of the entities in *V*, for example the size of the continuum. A monist thinks that even though our axioms have these shortcomings, our conception still refers to a unique and determinate object. A particularly strong kind of set theoretic pluralism denies this.

According to Hamkins (2012, 2015), the second part required to warrant the monists' quest for a new axiom, is what might be called a belief in uniqueness. Hamkins' primary focus is on undermining the assumption that there is a unique universe of sets. He argues that our experiences in studying different interpretations of ZFC and proposed axiom candidates over the past decades, which settles CH in different ways, give us reason to think that there is no single universe of sets. Instead there is a plurality of set theoretic universes, each with a corresponding conception of sets, all part of an existing set-theoretical multiverse. At least at first glance, the dialectical point is well-taken, set theoretical realism alone is not sufficient to establish monism about set theory.

Much of the motivation for the view comes from the work establishing that certain mathematical claims, like CH, are independent of ZFC. To do so set-theorists study so-called inner and outer models of ZFC, obtained by giving *possible* interpretations of the language of ZFC (for more detail, see Kunen 1980). Not all possible interpretations of ZFC are legitimately set-theoretic, for example I can take the domain to be all objects whatsoever and the membership-relation to be the identity relation. This is an interpretation of the language of ZFC, but without giving rise to any interesting set theory. Nevertheless, many of the possible interpretations do give rise to seemingly set-theoretic structures.

The inner models do not pose a particular problem for the monist, as we can think of such re-interpretations as restricting our view of *V* to only certain parts of it. According to Hamkins, the issue about the uniqueness of *V* arise from the construction of the so-called outer models, or forcing extensions, of *V*. The question is: do forcing extensions of *V* exist? Answering this question is not straightforward, given how the technical details of specifying forcing extensions of *V* from within ZFC are spelled out. We cannot prove from ZFC the existence of an outer model of ZFC (in fact, we cannot prove from ZFC the existence of any model of ZFC due to certain metamathematical results proved by Gödel), but we know how to speak, within ZFC, *as if* there were such an object, or in a way, to 'simulate' forcing extensions of *V* within *V* (Hamkins 2012:420). Hamkins thinks our mastery of this technique and the statements they result in, if read at face value, support the *existence* of such outer models, and thus universes beyond *V*. Make no mistake, this is controversial! But let us see how this might make trouble for the monist.

In sum, the pluralist thinks that much of this work should persuade us to think that there is no single *intended* interpretation for ZFC (which the monist presumes there is), but a range of legitimate interpretations. Furthermore, the *realist* pluralist claims that as far as the mathematics go, all of them exist. Arguably, this latter claim can be used in an argument against certain uses of extrinsic evidence, especially the use monists want to put it to.

Now, Hamkins himself uses a kind of inference to the best explanation-style argument, a kind of extrinsic evidence, in favour of pluralism, claiming that the multiverse conception better explains the experiences of theorists developing set theory through the last decades. This can be called a kind of higher-order use of extrinsic justification, in that it infers to a certain philosophical position to better explain our experience in mathematical practice. However, I think the truth of the higher-order ontological claim that there exists a multiverse blocks abductive reasoning at lower levels. Inferring to the absolute *truth* of a first-order set-theoretic axiom candidate because it has certain fruitful or verifiable consequences will usually not be legitimate if the multiverse actually exists. Of course, as I said, one might best establish the multiverse claim itself by abductive reasoning, so this is not a wholesale rejection of abductive reasoning in relation to sets. Still, if the argument for the multiverse view goes through, the use of inference to the best explanation (and maybe extrinsic evidence more generally) in support of a large class of interesting set-theoretic claims, that is, the axiom candidates, will be defeated by a certain ontological assumption about the subject matter. For any science, there must be a match between methodology and ontology. More particularly, if the multiverse exists, the monists' methodological approach fails to respect this.

Say you have two competing extensions of ZFC: ZFC+X and ZFC+Y. You are a realist, so accepting the one extension over the other is no trivial matter – it is supposed to be *true*. Say ZFC+X has a certain amount of extrinsic evidence in its favour, maybe it simplifies some difficult proof of an already established claim, or decides one of the independent statements of ZFC in which you have a prior interest, and so on. Maybe ZFC+Y has similar support, but let us say that it has less. Should this give you reason to think that ZFC+X is *true* as opposed to ZFC+Y? Given the truth of the multiverse claim, the answer is 'no,' because there might be set-theoretic universes where ZFC+Y holds as opposed to ZFC+X. The extrinsic factors might give you reasons for being particularly interested in 'ZFC+X'-universes, but they do not legitimize the further step of claiming X to be a basic principle of sets.

The idea of a multiverse is controversial, but let us state, in rough terms, the pluralists' positive view on the situation regarding CH. Hamkins (2015:137) thinks that for the most part CH is a settled question. Imposing his pluralist interpretation on the technical work exploring inner and outer models of ZFC, where on some CH holds and others ¬CH, he claims we have extensive knowledge about CH and how it behaves in the multiverse. The upshot is that there are several legitimate set *theories*, some of which entail CH and others ¬CH. Because of this, we shouldn't go looking for a unique solution for CH.

There are many challenges to this strong version of pluralism, some more philosophical, other technical. I will end with one in-between. We have seen (in section 2) that CH can be thought of as an issue in pure set theory (and here our intuitions about the nature of the stages beyond might not be very clear). But CH is also a claim in transfinite number theory (section 3), with potential consequences for how to think about the size of a very familiar mathematical object, $\mathbb{R}$. Thought of in this light, it will be problematic for some to hear that CH is supposedly both true and false, depending on what set theoretic universe you are talking about under a given interpretation. The set theoretic pluralist must convince anyone who thought they were referring to one, unique object, wondering about its specific cardinality, when trying to prove CH (or its negation), that she has been speaking ambiguously. Trying to talk about one thing in one way, when there are many.

5. Conclusion

In this essay we have been concerned with the notions of monism and pluralism in mathematics with focus on the truth-value of CH. Both views explored here have been realist about the existence of either a set theoretical universe or multiverse. To some this might be an odd way to frame the debate. It seems that especially pluralism is at least equally (maybe more) compatible with an anti-realist conception of mathematics. I leave that unexplored for now. Probably all four combinations are possible to defend. That is, monist-realist, monist-anti-realist, pluralist-realist, and pluralist-anti-realist. For example, one might hold a view that is pluralist on many ontological questions, yet have some notion of theoretical reason that might decide questions of truth.[8] At least in the *philosophy* of mathematics I suggest a working pluralism on these issues, so that we can explore the strengths and weaknesses of the possible positions more properly.

LITERATURE

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NOTES

[1] Ultimately, one defines the cardinal of a single set *A* to be the least ordinal α equinumerous to *A*, one can then prove the definition above. A cardinal number can be thought of as a special kind of ordinal, a so-called initial ordinal, which is an ordinal with the property of not being equinumerous to any smaller ordinal (for more see Enderton 1977: Chapter 7).

[2] The powerset of a given set *A* is the set of all subsets of *A*. The powerset axiom guarantees the existence of such a set for any set.

[3] This can also be stated, like Gödel (1947) does, as, for any α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. For the rest of the paper I will focus on the ungeneralized CH.

[4] To establish these results Gödel and Cohen developed powerful new formal techniques, working with so-called *inner* and *outer* models of ZFC, respectively. I return to this in section 4.

[5] I do not defend realism about mathematics in this essay, but point out certain connections between broader philosophical issues and mathematical methodology.

[6] This is not to say that they have no intrinsic plausibility. In certain guises the axiom of choice seems to have some intuitive appeal, for example the following formulation from Enderton (1977:151): Let *A* be a set such that (a) each member of *A* is a nonempty set, and (b) any two distinct members of *A* are disjoint. Then there exists a set *C* containing exactly one element from each member of *A*.

[7] This point refers to an issue with intrinsic appeal in mathematics, namely its inconstancy. The history of mathematics is full of claims changing status with regards to its perceived intuitiveness.

[8] Koellner (2009) defends something like this, in slogan form 'existence in mathematics is cheap, truth is not.'