WHAT NUMBERS MIGHT BE

ALL MATHEMATICS ARE COMPRISED BY A SINGLE INFINTE STRUCTURE

Structuralism is the view within philosophy of mathematics that the true nature of mathematics is a study of structures representing relations between mathematical objects. In this paper I will introduce my own branch of structuralism, structuralism B, where I propose any mathematical structure to be a part of, and dependent on, an infinite all-encompassing structure. I also discuss structuralism B in connection to Kuhn's scientific revolutions, I argue that mathematics behave differently than the natural sciences with more of a cumulative increase in the understanding of its subject matter as opposed to the revolutionary jumps we find in Kuhn.

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What is a number? Say we focus on a single number, a 2. We seem to have a clear mental image of what a 2 might be. If you have one orange, and then another orange, you would have two, 2, oranges. This also explains the expression 1 + 1 = 2 and it pictures how we could imagine the historical need to make the symbols "1", "2", "+" and "=." Still the image we have will often be of objects in a quantity of 2 or as the mathematical symbol "2," the meaning of the number does not appear "on its own." It seems strange that we can have such a clear mental image of a 2, but yet its meaning seems to be hidden within other "objects." Is the 2 dependent on countable objects? Is it dependent on its predecessor? Does the 2 exist independently of the mathematician? Are there different kinds of 2's?

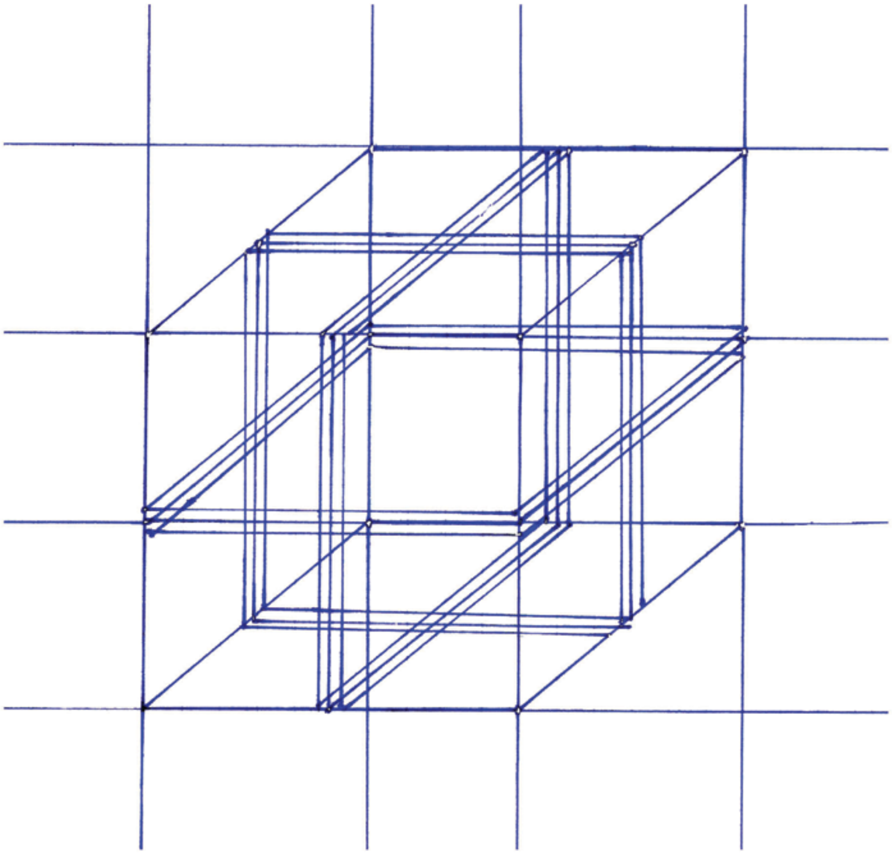
It can be useful to begin by introducing some terminology. *Realism* is the claim that some object, dependent on what kind of angle you would want to claim realism from, exists independently of beliefs, language, culture, and so on (Miller 2016). *Realism in ontology*, we define as the view that (at least some) mathematical objects exist independently of the mathematician (Shapiro 2000). In opposition, we have views like *idealism* and *nominalism*, where the former agree that mathematical objects exist but say they are dependent on the human mind and the latter reject the mere existence of mathematical objects altogether.

Another definition we will need is the *type-token* distinction. How many words are there in "dog dog dog"? There is one word *type*, but three *tokens* of that type (Papineau 2012). In addition, for those who have not had a close relationship to mathematics in a while, I also include the explanation of the term *natural numbers* as the numbers we can count: (0,) 1, 2, 3, … There is to this day no agreement in the mathematical community as to whether or not 0 should be counted as a natural number.

STRUCTURALISM: 2 IS A POSITION IN A STRUCTURE

Structuralism is the view that mathematics is the science of structure (Shapiro 2000). In talking about numbers, this essentially means that the nature of the number lies in its relation to other numbers, not in the meaning of what a number is on its own. Benacerraf (1965) stated that it is meaningless, in his words "pointless to the extreme" (Benacerraf 1965:290), to question whether the number 3 is the same as a particular *object* or not, like $\{\{\{\emptyset\}\}\}$, the set theoretical Zermelo definition of 3 (more on this later). What matters is the abstract structure that numbers are the elements of; the natural number 2 is a position in a structure (Resnik 1997). In other words, structuralism denies any ontological independence of the individual number. We have chosen to focus on numbers, but this does not mean that structuralism limits itself only to number structures. Resnik (1997), as one of the most important figures in developing structuralist views, has approached structuralism with a background in geometry and model theory.

Shapiro (2000) defines a *system* as a collection of objects that have relations between them, and a *pattern* or a *structure* to be the abstract form of this system. There are different branches of structuralism, but there seems to be an agreement among them that there can be different systems that exemplifies the same structure, which makes



a structure one-over-many. As an example, the relations between oranges in a pile will exemplify the natural number structure, they can be counted as 1, 2, 3, and so on. The relationship between a structure and a system *being* structured is analogous to the type-token distinction mentioned earlier. A token is a physical object that can be destroyed (Shapiro 2000), a type is abstract, and there can be more tokens for one type as there can be more systems that exemplify one structure.

Structuralists agree on the denial of any ontological independence of the individual position in a structure, in other words, the single number does not exist on its own. Does the structure itself exist independently of the system? We distinguish *ante rem* from *in re* structuralism. Ante rem structuralism is the realist view that the structure does exist independently of the system, in re or in rebus (Horsten 2016) structuralism holds that the structure exist, but only *in* the system. There are also nominalist views that the structure does not exist beyond language.

An objection to ante rem structuralism, and any form of realism in ontology, is sometimes referred to as Benacerraf's *identification problem* (Horsten 2016). We present the problem as follows: in set theory, a mathematical theory about *sets*, or, informally, collections of things, there are two different commonly used definitions of the natural numbers, named after the set-theoreticians Ernst Zermelo and John von Neumann. They both derive the natural numbers from the empty set, a "collection of nothing," noted Ø, which the two definitions also both define as 0. Without explaining this any further, let us look at {{Ø}} and {Ø, {Ø}}, the two different Zermelo and von Neumann set theoretical definitions of the natural number 2, respectively. Benacerraf (1965) asks: How can we know which one of these is the *true* 2?

Shapiro (2000) argues that in structuralism, this question needs no answer: "A natural number is a place in the natural number structure. The latter is the pattern common to all of the models of arithmetic, whether they be in the set-theoretic hierarchy or anywhere else." (265).

As long as the given definition of 2 can be shown as a position in a structure that satisfies the laws of arithmetic, it is a valid description of the place 2 in a structure. A green unicycle is not a place in a structure, therefore unicycle is not a number.

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Another question that originates from Benacerraf is how we can have knowledge of structures. If the structures are abstract entities, how can we know about them? This is sometimes referred to as the epistemological challenge. Resnik (1997) answers that he hypothesizes our ancestors by using written diagrams and design patterns assumed geometric objects *sui generis*, in a class by itself. From then on it was relatively easy to add other mathematical structures to those already known. Still, not everyone agree on this, and Benacerraf's ideas have for some been a reason to dissociate themselves from realism in ontology.

STRUCTURALISM B

My view on the nature of mathematics relies heavily on structuralistic ideas, but I want to specify that I believe that we are talking about *one* structure, not a collection of smaller structures independent of one another. This one single structure, or pattern, is the relations of all (pure) mathematics, and as I am about to explain, my view is that we do not, and will most likely never, understand the limits and complexity of this structure. I call this view s*tructuralism B*.

Structuralism B, like other structuralist views, holds that the meaning of mathematical objects lies in their positions or places in a structure, but it seems to differ from other structuralist views in that we are talking about a *monadic* structure, it is single, and not *polyadic* consisting of many smaller independent structures. Shapiro states: "The number 2 is no more and no less than the second position in the natural number structure" (2000:258). I want to argue that it is not as simple. The number 2 is more than the second position in the natural number structure. Firstly, the monadic structure means that the natural, the integer, rational, real and complex number 2, +2, 2/1, 2.000…0 and $2 + 0i$ is unambiguously and clearly the *same* number. That the numbers are different sets in set theory merely says that the same number can be expressed as different sets. Further, these numbers that in our given mathematical language can be referred to as different *kinds* or elements in different sets of numbers, or different sets altogether, are what connects the structure together. The complex numbers are for example connected, say to the real numbers in the structure through the numbers within the complex numbers that can also be regarded as real, that is numbers on the form $a + bi$, where $b = 0$. This adds some more complexity to the traditional structuralist answer to what a 2 might be. A 2 is a position in the structure, but it is not sufficient to say that it is the second position as this seems to only be related to the natural numbers. It is also the +2nd, 2/1nd, 2.000…0nd and $2 + 0i$nd. The structure does not only represent the relations of numbers within a specified set, it also represents the relations of the different sets of numbers. This is a crucial idea in the philosophy of structuralism B.

Structuralism B is an ante rem type of structuralism, it implies that the mathematical structure is real and exists independently of the mathematician. It is important to specify that structuralism B, like any other ante rem variation of structuralism, does not claim independent existence of the places or positions in the structure, in our case numbers, only ontological independence of the structure itself. *Unlike* other views of structuralism, structuralism B does not claim any ontological independence of any *part* of the structure by itself either, here for instance what Shapiro (2000) would refer to as the natural number structure. The natural numbers do not exist independently without the real, the complex, and so on. This might seem as a controversial claim: Does this mean that the natural numbers we use when counting objects in the real world are nothing by themselves? The answer is yes and no. The natural numbers are in no doubt expressing relations in the real world, which is very useful, but in counting, say oranges, we only use a small fraction of the structure. As it is as natural for us to count and to cut up oranges in smaller pieces, we use the natural and rational numbers and see them as a part of a bigger whole.

It seems to me that there is no doubt about the independent existence of the structure itself. Just because this is not something we can *sense* by itself, without objects, does not rejects its authenticity and independence of the mathematician. A 2 is only visible for us in the case where we have two objects, like a wave and its mathematical properties is only visible to the naked eye in the case where we have water.

According to structuralism B, although the mathematician did not make the mathematical structure, the mathematician *did* make the mathematical system we use. An example of this is that our decimal system most likely is a result of counting on our ten fingers. There are reasons to believe that it would have been easier if we counted in eights or twelves instead of tens. Essentially, we could have chosen to develop a different mathematical system, but it still would have explained the same mathematical structure. A mathematical system exemplifies the structure. I believe our mathematical system could be different, as long as it would instantiate the relations between mathematical objects.

STRUCTURALISM B IN LIGHT OF KUHN'S SCIENTIFIC REVOLUTIONS

Another important idea of structuralism B is that our understanding of the structure is seemingly forever expanding. We keep adding to it with mathematical results representing the relations of the structure.

We might think of our epistemic relationship with the structure of structuralism B like us being blind in an infinitely large furnished room. Blinded, we cannot see a complete overview of the furniture and the relations between them, so we stumble around in the dark to find and identify the furniture piece by piece. In the same way, we have historically discovered mathematics, gradually revealing part by part of its structure. As mentioned earlier, we can imagine the need for making the symbols to express that one plus one orange equals two oranges. In other words, it is easy to imagine how the natural numbers were *discovered*. At some point, there was a need for a symbol for nothing, 0, and then, in situations of debt, the negative numbers were introduced. It is said that the Pythagoreans were the first to discover irrational magnitudes (Huffman 2014), and in relation to area of circles and the hypotenuse of an isosceles triangle with catheti size one, it later became necessary to express numbers like π and $\sqrt{2}$. As of today there is no good way of expressing the exactness of π in our decimal numeral system. In the discovery of practical use of the square root of negative numbers, there was a need to introduce $i = \sqrt{-1}$, our number system was not sufficient enough. Like the blind discovering furniture piece by piece in the infinite room, we have been discovering and extending our perception of the mathematical structure, number by number. That we chose to use the decimal numeral system somewhere along the way is not even the best way to demonstrate the structure. Similarly, algebra has been developed by adding definitions as results were found where these definitions were needed. As the world of mathematics is expanding, and as mathematicians keep adding results, the knowledge we have of the structure is too.

Thomas Kuhn (1922–1996) is said to have written the most famous book about science in the twentieth century, *The Structure of Scientific Revolutions*, first published in 1962. Briefly, it states that normal science is always practiced within a *paradigm*, a way of doing science that includes specific methods for analyzing data and habits for scientific research. When *scientific revolutions* occur, they are triggered by *anomalies*, phenomena that cannot be explained in the existing paradigm, and as a result the paradigm is changed. I include this as I see similarities between the historical development of our mathematical system and the change of paradigms in natural sciences. Does the development of mathematics represent paradigm shifts, as according to Kuhn?

The answer is no, not in the same way as in the (other) sciences. I include a quote from Shapiro that explains the uniqueness of mathematics as a study:

> Unlike science, mathematics proceeds via proof. A successful, correct proof eliminates all rational doubt, not just all reasonable doubt. A mathematical demonstration should show that its premises logically entail its conclusion. It is not possible for the premises to be true and the conclusion false. (2002:22)

Mathematics can be proven, hence in my view it has an objective truth-value. In a way we can see mathematics as being done within paradigms, before knowing about the imaginary numbers, mathematics were done as though they did not exist. An important difference is nevertheless that we do not discard the old paradigm, we seem to build on top of it. Discovering new entities within mathematics has expanded our *understanding* of the structure, not expanded the structure itself.

There is no reason for us to assume that our system is complete. Looking at mathematical history, there has been a constant need of new discoveries to prove our statements derived from the system we have been working in. We cannot fully perceive the structure in the system we are using. There is a chance that we can never fully understand the structure, and that our understanding of it will always remain incomplete.

There is one last open question that can be derived from structuralism B. If the study of mathematics and our understanding of the structure have been developed because we needed proofs of statements we could not find in our given system, this must mean our results came from us looking for it. Does this mean that the mathematician discover only what they seek? Can there be further discoveries with a different approach? We save that for another time.

LITERATURE

Benacerraf, P. 1965, "What Numbers Could not Be," in Benacerraf, P. & Putnam, H. 1983, *Philosophy of mathematics*, Second Edition, Cambridge University Press, Cambridge.

Horsten, L. Winter 2016, *Philosophy of Mathematics*, The Stanford Encyclopedia of Philosophy, E.N. Zalta (ed.), viewed 1 December 2016, forthcoming URL: <http://plato.stanford.edu/archives/win2016/entries/philosophy-mathematics/>.

Huffman, C. Winter 2016, *Pythagoreanism*, The Stanford Encyclopedia of Philosophy, E.N. Zalta (ed.), viewed 22 November 2016, forthcoming URL: <http://plato.stanford.edu/archives/win2016 entries/pythagoreanism/>.

Miller, A. Winter 2016, *Realism*, The Stanford Encyclopedia of Philosophy, E.N. Zalta (ed.), viewed 22 November 2016, forthcoming URL: <http://plato.stanford.edu/archives/win2016 entries/realism/>.

Papineau, D. 2012, *Philosophical Devices*, Oxford University Press, Oxford.

Resnik, M.D. 1997, *Mathematics as a Science of Patterns*, Oxford University Press, New York.

Shapiro, S. 2000 *Thinking about mathematics*, Oxford University Press, New York.