SET THEORETIC REALISM AND LINGUISTIC FRAMEWORKS

How should we understand statements like "set are objects" and "numbers are properties of sets? Drawing on Rudolf Carnap's theory on linguistic frameworks in "Empiricism, Semantics and Ontology," I shall attempt to answer this question. The discussion will be centred on a view espoused by Penelope Maddy in the two articles "Perception and Mathematical Intuition" and "Sets and Numbers," known as set theoretical realism. I argue that this view can best be understood as a linguistic framework in itself, and can plausibly be extended to a framework I call generalized scientific realism, a framework in which statements like "sets are objects" and "quarks are objects" have the same status.

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Puzzling claims

"Sets are objects," "numbers are sets" and "numbers are properties of sets." These and other related phrases have for a while struck me as quite odd. On a surface level, they seem perfectly acceptable. Considering the first example, for instance, it might seem obvious what is meant: Books are objects, automobiles are objects and sets are objects. Like books and automobiles, sets exist 'out there' in the external world, independently of human beings that theorize about them. Sets can have properties, like 'size' and 'ordering,' which they share with some other sets, but not with others.

Yet upon further reflection, this account becomes less intuitive. In many ways, sets are not like objects at all: Most things I count as 'objects' have extension, mass and colour, and I can interact with them by touch or smell or sight, i.e. they inhabit the physical world, and have physical properties. Sets – the kind of things described by set theory[1] – seem very unlike many of these paradigmatic instances of 'object.' Sure, sets can have properties, but so can 'love,' 'a sentiment' or 'red,' and I am not sure whether I am inclined to say that any of these things are objects.

An immediate response to this sort of argument is that sets are a special type of object; not a *physical* object, but an *abstract* object. Though not a part of the physical world, they still share the common property of 'objecthoood' with the physical objects. This does not lessen my confusion about what is meant by 'sets are objects': Having no previous experience with abstract objects, what I am able to take from this explanation is that sets are like physical objects, except in every way that makes something a physical object. To me it seems like the meaning of 'object' has shifted from how I usually understand it, leaving me none the wiser as to what is meant by 'sets are objects.'

In an effort to gain greater understanding of what is meant by locutions of this sort, I shall discuss Penelope Maddy's *set theoretic realism* in light of Rudolf Carnap's theory on lingustic frameworks in "Empiricism, Semantics and Ontology" (Carnap 1950). It is my hope, that by drawing on the methodology and distinctions Carnap outlines in this article, I will be able to clear up the status of Maddy's claims, which are of the sort that I find puzzling. In the process *set theoretic realism* will be interpreted as a linguistic framework – within which it is possible to answer questions of the form "are sets objects?" – and so the merits of this particular framework will be discussed.

Carnap's theory of linguistic frameworks

The key idea in "Empiricism, Semantics and Ontology" is that new entities – objects, properties, numbers or whatnot – are introduced into language by the creation of new lingustic frameworks. In creating a new linguistic framework, we "introduce a system for new ways of speaking, subject to new rules" (Carnap 1950). Furthermore, questions about the existence of a particular entity can only be asked within some framework, i.e. they are *internal* questions. As examples of such frameworks, Carnap discusses *the world of things*, *the system of numbers* and *the system of thing properties*. Similarly, when we discuss literature, we adopt lingustic frameworks where statements like 'Sherlock Holmes and John Watson are friends' or 'Harry Potter's boggart takes the form of a dementor' can be evaluated. Questions about which framework one should accept are then settled on purely pragmatic

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grounds. When these *external* questions are not construed as being about the usefulness of a particular framework, but as questions about the reality of the particular framework, they become meaningless *pseudo-questions*; there is no way to settle a disagreement on the reality of a particular framework without first adopting another framework within which such questions become internal questions (Carnap 1950).

The last point – i.e. that questions about the reality of the framework are pseudo-questions – is the more controversial of Carnap's claims, and in choosing his theory as a starting point I might be criticized for just ignoring the entire issue. On this view, questions like "are sets objects?" or "are numbers properties of sets?" might seem like just the sort of pseudo-questions that Carnap dismisses, at least if they are understood in the sense of 'is the correct system one in which sets are objects?' These are the very sort of questions that are debated by realists and anti-realists, and to simply dismiss them would be to shy away from the discussion entirely.

Fortunately, I do not need to commit myself to this stronger claim in order to find some use of Carnaps theory: Drawing on the idea of lingustic frameworks, I can evaluate claims of the sort 'sets are objects' and 'numbers are properties of sets' as internal to a particular framework. Furthermore, the internal questions about the objecthood of sets and the propertyhood of numbers can be separated from questions about accepting the framework of set theory or the framework of mathematics. Hence, I can discuss the claims by Maddy in her exposition of set theoretic realism without making a judgement on whether this is the 'correct' framework.

Set Theoretic Realism

In the papers "Sets and Numbers" and "Perception and Mathematical Intuition" (1980; 1981), Penelope Maddy develops the position she calls *set theoretic realism* as she responds to some challenges put forward by Paul Benacerraf against different realist interpretations of set theory and matemathics. Maddy's position is based on the idea that set theoretic realism/mathematical realism is analogous to more ordinary forms of scientific realism:

> The science of mathematics is strictly parallel to physical science; both begin from perceptual and intuitive bases (realism$_2$), and grow by the additional layers of theory which are not justified perceptually or intuitively, but theoretically (realism$_3$). (Maddy 1981:497)

On this view, similarily to how we have pre-theoretic, perceptual knowledge about the physical world, we have pre-theoretic knowledge about sets and mathematical entities. From this basic knowledge about sets and mathematics, we build up our mathematical theories, which are justified in the same way as our physical theories; by merits like accuracy, predictive ability, explanatory power and scope (Maddy 1981:498).

Furthermore, we gain pre-theoretic knowledge about sets concurrently with our perceptual knowledge about physical objects (Maddy 1980:178–79): When I perceive the books in my bookshelf, I not only perceive each book individually, but also the set of books. As I perceive the set of books, I also form some simple intuitive beliefs about this set and sets like it. Maddy suggests that some of these beliefs might be expressed as "sets have number properties," "any property determines as set of things which have that property," "the number property of a set is not changed by moving its elements" (Maddy 1980:185).

The motivation of this account is to argue that the mechanism that produces intuitive beliefs about sets is no more mysterious or strange than the mechanism that produces perceptual beliefs about physical objects. Here, Maddy relies on a theory of perception in which our perceptual apparatus does not merely passively receive information from the senses, but also gives the information structure, and we gain perceptual beliefs about physical objects through this process.

The most obvious response to this argument is that to take the proposed analogy seriously, perception of mathematical objects and mathematical intuition must be included and studied within the domain of the natural sciences. Maddy's views on perception are justified since there is a scientific discipline – perceptual psychology – tasked with studying our perception of physical objects, which we lack in the case of mathematical intuition. The latter claim might be challenged, though: Cognitive psychology is tasked with studying our ability/predisposition to place objects in different categories, and to identify the relevant distinctions between the categories (Maddy 1980:180–81). This might be considered as part of a study of our perception of sets, and our intuitive beliefs about these sets. The ability/predisposition to organize the world into categories has been acquired through evolutionary pressure that has shaped our brains to organize information in this particular way, just like evolutionary pressure has shaped our brains to process the two-dimensional pattern on our retinas as a three-dimensional model of the world. Both our ability to process two-dimensional patterns as three-dimensional images and our ability to create and reason about categories have been conducive to our survival as a species.

The case for what Maddy calls *intuitive beliefs* is flimsier, as the types of beliefs expressed by claims like 'any property determines a set of things which have that property,' are quite complex, and thus difficult to study from the viewpoint of cognitive psychology. There is less evidence that we have cognitive mechanisms appropriately linked to these "intuitive beliefs," and also that these beliefs are the ones expressed as 'our' mathematical intuitions. Yet, the complexity does not rule out the possibility that we can develop a scientific field that studies our mathematical intuitions just like perceptual psychology studies our perceptual apparatus.

At this point, we have the outline of the set theoretic realism that Maddy advocates, and are ready to address some of her more specific ontological claims: To Maddy, sets are objects, the simplest of which are sets of medium-sized physical objects (Maddy 1981:497). Moreover, there is a relationship between numbers and sets. This is important since, in some cases, the theoretical justification we have for axioms of higher set theory is their explanatory power in number theory. For the proposed analogy between scientific realism and set theoretic realism to work, we must give some account of the relationship between numbers and sets (Maddy 1981:499).

One straightforward way to establish this relationship, is to simply claim that numbers are certain sets (Maddy 1981:499). However, accounts of this type will run into the ontological challenge raised by Benacerraf in "What Numbers Could not Be," namely "if numbers are sets, what sets are they?": Since numbers can be expressed as sets in multiple ways, and there is no argument to settle which of the different possible accounts is correct, then there is no 'correct' account that identifies the numbers with particular sets. From this, Benacerraf concludes that the numbers cannot be sets (Benacerraf 1965:62). Later, he extends this argument to purportedly show that numbers cannot be any other sort of object either, since all objects that form a recursive progression is adequate to give an account of numbers (Benacerraf 1965:69–70).

In response to this argument, Maddy concedes that numbers are not certain sets, but rather properties of sets (1981:507). The number property of sets is analogous to physical properties like length, and mathematics is the discipline where we study such properties just like physics is the discipline where we study physical properties. Maddy further explains the possibility of expressing sets in multiple ways as corresponding to different rules for measuring the numbers' properties (1981:504).

To sum up Maddy's views: Mathematical/set theoretic realism is not that different from scientific realism in general; mathematical science works similar to the physical sciences; we perceive sets and gain intuitive beliefs about sets concurrently with perceiving physical objects and gaining perceptual beliefs about them; sets are mathematical objects, the simplest of which – the ones we first gain access to – are sets of medium-sized physical objects; and the numbers are properties of these objects. The main thrust of this line of argument is to expel the mystery of mathematical realism; to show that it is no more mysterious than 'less controversial' realisms.

As I see it, Maddy is largely successful in this endeavour: Though one might be sceptical that there are cognitive mechanisms that give rise to "intuitive beliefs" about mathematical objects, thus breaking the analogy somewhat, Maddy makes a compelling case that set theoretic realism is not that different from, say, scientific realism. Of course, it then also shares the same merits and problems.

Answers to the internal questions

We are now ready to move on to the main question of this paper: How should we evaluate claims like 'sets are objects' and 'numbers are properties of sets'? With Carnap's distinction between internal and external questions in mind, we can first try to establish what category these claims fall into: Do they mark an acceptance of a particular framework, or are they assertions internal to the framework in question?

First, we may note that none of these claims directly address the usefulness of a given framework. One alternative, then, is that they address the question about the reality of a certain framework, i.e. what Carnap would consider a pseudo-question. The assertion would then be of the form 'the framework in which sets are objects is the correct representation of reality.' This would introduce more complexity than we started with, since we must include new entities like 'framework,' 'correct,' 'representation' and 'reality' and rules to evaluate statements about these entities. At this point, we do not have the resources to evaluate claims about the reality of the framework. A more fruitful venue might then be to evaluate claims like 'sets are objects' within a particular linguistic framework, i.e. as addressing an internal question. We then need to figure out what framework this is.

A suitable place to start is what Carnap calls *the world of things*, i.e., the framework which contains the entities we deal with in everyday language (1950), in which we carry out empirical investigations to discover the answers to internal questions like "is there any tea left in the cup?" and "has anyone tried to contact me?", and which contains words like "tea," "cup" and "phone". More importantly, it contains the word "object" inasmuch as "a cup is an object," "a chair is an object," "a sentiment is not an object" occur as sentences in the language, so this is promising. I shall use "$\text{object}_{\text{WoT}}$" when talking about "object" as it occurs in thing-language.

How should we evaluate the statement 'sets are $\text{objects}_{\text{WoT}}$'? According to Carnap:

> To recognize something as a real thing or event means to succeed in incorporating it into the system of things at a particular space-time-position so that it fits together with other things as real, according to the rules of the framework. (1950)

In other words, whether sets are $\text{objects}_{\text{WoT}}$ within thing-language will depend on whether we are able to incorporate sets in the system together with the other $\text{objects}_{\text{WoT}}$. This requires us to add the word "set" to the thing-language, but this is not really a problem; we come up with new words to describe the world of things all the time.

The most obvious problem is that sets don't have a space-time-position, unlike physical objects. Maddy argues that some sets – the sets of physical objects – are in fact spatio-temporally located; they are located where the aggregate of the $\text{objects}_{\text{WoT}}$ are located at the time the $\text{objects}_{\text{WoT}}$ exist (1980:179). However, as she also admits, some sets clearly aren't, and thus we would need to extend our framework to allow for some non-located $\text{objects}_{\text{WoT}}$. Similarly, claims that are true relative to the ordinary thing-language, e.g. '$\text{objects}_{\text{WoT}}$ have extension,' '$\text{objects}_{\text{WoT}}$ obey physical laws' must be revised. Without these revisions, sets are more similar to $\text{non-objects}_{\text{WoT}}$, like sentiments (which are neither located nor extended, and do not obey physical laws), than to $\text{objects}_{\text{WoT}}$. Without extending and revising thing-language extensively, we must conclude that *within thing-language*, sets are not $\text{objects}_{\text{WoT}}$.

Another possible framework is the framework of set theory itself. However, here the problem is with the word "object": It does not occur within set theory. We could do as in the case of thing-language, and just add the word to the language. However, we accomplish very little by this addition, as this does nothing to change the theorems and axioms of set theory, and thus does not affect the system in any interesting way.

Similarly, for 'numbers are properties of sets': Here, we need a framework which contains words like "property," as well as "numbers," variants of number words and "sets." A starting point might be what Carnap dubs *the system of thing properties*, extended so that it includes words not occurring in thing-language, such as "set" and "number," as well as general terms like "property" and variables like "f" (1950). Provided that you have found some way of including "sets" in the thing-language, then, you can at least form sentences of the form "the set of books on the table is three-numbered" in which the numbers occur within the framework as properties of sets. Note, however, that this does not apply generally, only to sets of physical objects and thus finite numbers. To extend this so that numbers *in general* are properties of sets *in general* involves a further extension of the framework.

Then what *is* the linguistic framework in which sets are objects and numbers are properties of sets? The answer might seem surprisingly straightforward: It is the framework of set theoretic realism, which, if Maddy's argument goes through, is extended to a generalized scientific realism. Choosing "$\text{object}_{\text{STR}}$" and "$\text{object}_{\text{GSR}}$" for the word "object" as it occurs in the framework of set theoretic realism and generalized scientific realism respectively, we can assert 'sets are $\text{objects}_{\text{STR}}$' and 'sets and quarks are $\text{objects}_{\text{GSR}}$.'

I am here using "framework of set theoretic *realism*" and the "framework of scientific *realism*," as opposed to "the framework of set theory" or "the framework of (for instance) physics." This is because the framework of set theoretic realism contains more than the framework of set theory: It, for example, contains the word "object." There are other frameworks compatible with set theory, where it is not the case that sets are mathematical objects; sets might be mental events, symbols or a particular Matryoshka doll. Similarly, there are frameworks that are possibly compatible with our current scientific theories, but where electrons and genes aren't spatio-temporally distributed objects, but rather complexes of sense-data. What I am considering here is the particular interpretation of set theory which Maddy advocates for, in which sets are mathematical objects playing a similar role in mathematics as the physical objects do in the physical sciences.

To include "sets are $\text{objects}_{\text{GSR}}$" within generalized scientific realism, where you also include things like quarks and electrons as $\text{objects}_{\text{GSR}}$, might be a bit tricky though. Here you need a general notion of '$\text{objecthood}_{\text{GSR}}$,' as well as ways to discriminate physical $\text{objects}_{\text{GSR}}$ subject to the laws of physics from the mathematical $\text{objects}_{\text{GSR}}$, which need not be constrained by these laws. In such a framework, it might be possible to start from statements like "there is a set of books in my bookshelf," build up towards more complicated theories like quantum field theory, and along the way find a place for the notion of non-physical objects_{GSR}, like sets.

Establishing this link between mathematics and physical theory would make mathematics subject to the same empirical investigations as any physical theory, and therefore take some 'purity' out of it. The answers to questions like "is the sum of the angles of a triangle 180°?" or "is the cardinality of the real numbers the smallest uncountable cardinal" would in part be determined by empirical investigation (as long as empirical investigation continues to be a part of our mathematical-physical theories): If the continuum hypothesis is true, it will be a part of a complete description of mathematical-physical reality; if it is false, it will not. Thus, if it is integral to our best scientific theories, we have reason to accept it; if it isn't, we don't. One benefit of this framework is that – within it – 'property' will be univocal in "energy and momentum are two properties of physical objects" and "numbers are properties of sets." However, this relies on whether Maddy's proposed analogy between the mathematical and physical sciences works.

Note that these claims are made within our hypothetical framework of a generalized scientific realism; there might be other frameworks within which the questions have different answers, or maybe no answer at all. I have yet to claim that this framework is 'correct,' and no such claim will be made here. I have only tried to show how to make sense of claims like "sets are objects" by evaluating them within a linguistic framework, and shown how it might be possible to make sense of such claims by evaluating them within the framework of set theoretic realism as it is presented by Maddy.

Though I agree with Maddy that set theoretic realism is no stranger than run-of-the-mill scientific realism, and that set-theory, mathematics and the physical sciences might best be considered as integrated in a single master science, I am not inclined to assert the correctness of this framework. Like Carnap, I am more interested in the usefulness of the particular framework than whether or not it is 'correct.' Someone more inclined towards realism than myself will possibly disagree with me at this point. So be it – this is not the place to settle the age-old debate on realism vs. pragmatism.

Conclusion

I introduced this paper by expressing my puzzlement over statements like "sets are objects" and "numbers are properties of sets." Now, the source of that puzzlement is clear: In order to evaluate such claims, one must first specify a particular linguistic framework, which in the case of these claims isn't obvious from context. The acceptance of these claims would involve adopting a framework like Maddy's set theoretic realism. Maddy has argued – convincingly, I think – that this is no more problematic than to accept ordinary scientific realism. In fact, if we follow Maddy's further arguments that mathematics and the physical sciences can best be seen as integrated in a single master science, then set theoretic realism is really a continuation of scientific realism into a generalized scientific realism. If this is the case, then it admits the possibility of a framework in which both sets and quarks are objects_{GSR}.

Though this might make the generally realist philosopher less wary about accepting sets into their ontology (or maybe warier about accepting scientific realism?), this does nothing to settle the realist-anti-realist-debate. I am not even sure whether it even can be settled, as I am unable to meaningfully articulate the ontological concerns in question, much less find a way to settle them. Nevertheless, this does not stop us from accepting a linguistic framework for pragmatic reasons alone.

Should we, then, as Maddy advocates, accept a generalized scientific realism? I am not sure. Maddy has argued that it is at least possible in principle, but whether it is possible in practice is another matter. Part of this effort would be to make our mathematical faculties and intuitions part of a naturalistic study, just like cognitive psychology studies our perceptual apparatus. At that point, we would be on the way towards a fully integrated master science.

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NOTES

[1] Informally, set theory is the mathematical theory of collections, known as sets, whose members are elements of the sets. In pure set theory, all the elements are themselves sets. It is possibly to reduce parts of higher-level mathematics, like arithmetic, to set-theory.