PLATONISM, REDUCTION AND THE AXIOMS OF SET THEORY

Set theory taken at face value refers to and says things about abstract objects, i.e. pure sets. If, however, sets are abstract and unable to causally interact with us, then how could we know which axioms correctly capture what sets exist and how they behave? In this paper I look at the relationship between Platonism and justification of axioms of set theory. I present an argument that supports adopting the Axiom of Choice as an axiom of our set theory. The main goal of this argument is, however, not only to support this conclusion, but also to demonstrate how one can motivate axioms by abductive reasoning when assuming Platonism. The core of the argument is reductionism, namely that all of mathematics can be reduced to set theory. The prospects of reductionism can in turn motivate axioms that enable this.

By Max Johannes Kippersund

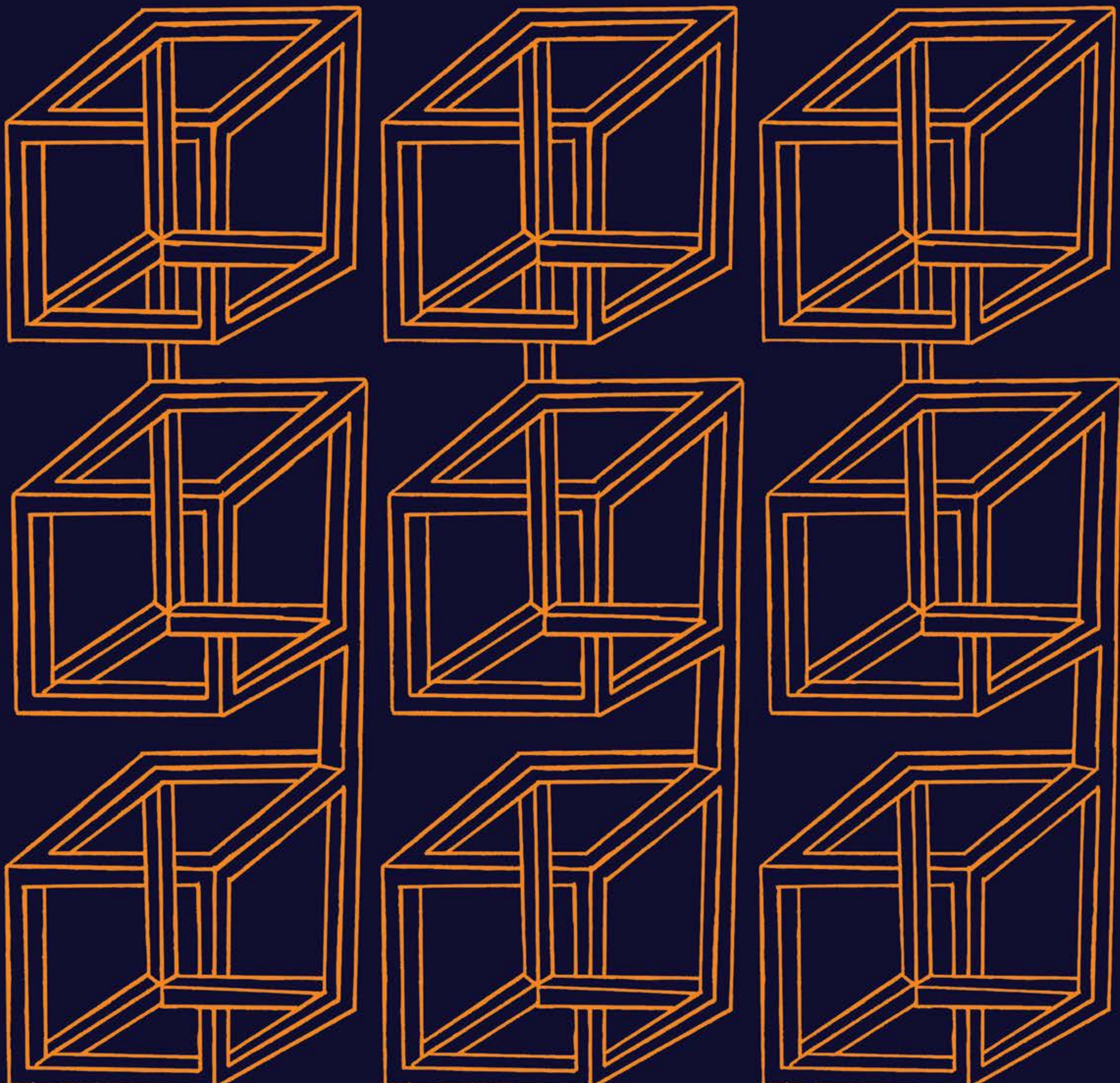
In the philosophy of mathematics one is often presented in various forms with the idea that set theory provides the "logical foundations for arithmetic and analysis" (Zermelo 1908:200), has a role as a "backdrop for classical mathematics" (Maddy 2011:33) or that the vast body of modern mathematics can be deduced from our axioms of set theory (Horsten 2016). These are in themselves intriguing claims in need of further explanation. They also, however, raise the more pressing question of what contemporary set theory itself is founded on. Just as often as the foundational status of set theory is heralded, one is also confronted with stories of inconsistency, like how Frege's logicist project was thoroughly derailed by Russell's paradox. What reasons do we have for believing the currently accepted axioms of set theory?

In this paper I look closely at how a platonist can defend axioms of set theory in responding to Benacerraf's epistemological challenge. This is the motivating background of the paper. I then look closely at Penelope Maddy's distinction between intrinsic and extrinsic justification. I point out that intrinsic justification is very much compatible with a platonist ontology, but not very illuminating when it gets to the epistemological challenge.

I then turn to the main part of this paper which is an original and abductive argument for the Axiom of Choice as a true axiom of set theory. This argument ties several influential lines of reasoning together. It rehearses the well-known indispensability argument for platonism. It also demonstrates how a platonist about mathematics can have concerns regarding parsimony in ontology. And how this in turn can combine with the idea that set theory provides the foundations for other branches of mathematics, in order to motivate the view that axioms that make set theory strong enough for the *reduction* of all of mathematics to set theory are true.

This argument aims to show three things, first, and most directly, that its conclusion is correct; we should adopt the Axiom of Choice (AC from now on) as an axiom of set theory. This is not intended to be a controversial conclusion. Most mathematicians and philosophers do, after all, accept AC by now. The perhaps more interesting philosophical points are the other two, which are indirectly demonstrated against the motivating background of the paper: Firstly, that a platonist can draw on a whole range of considerations when justifying axioms, for instance abductive reasoning, and secondly, to illustrate a form of anti-exceptionalism about foundational issues in mathematics.

Illustrasjon: Abirami Logendran



Platonism and Set Theory

If we take the terms and sentences of set theory at face value, they refer to and say things about sets. But what are sets? They do not seem to be concrete, like physical objects. Although one can reason about sets of such objects, e.g. the set of my phone and my favorite book, the sets mathematicians most often talk about are the so-called *pure sets*, namely, sets with only pure sets as members. That is, all members of a pure set are sets, any member of a member is a set, any member of a member of a member is a set, and so on. These objects are, I think, correctly called abstract. In taking sets, and other things mathematicians talk about, to be abstract objects existing independently of us we are adopting a platonist view. I take abstract here to apply to objects that do not enter into causal relations, are eternal, unchangeable and not located anywhere in space and time (similar to how Stewart Shapiro understands it, 2000:27). Taking mathematical language at face value is the most straightforward reason for arriving at platonism (this is suggested by Shapiro 2000:25, Benacerraf 1973:403); we do not know that numbers exist from direct observations of them, but we might think they exist because a reasonable semantic theory applied to mathematical language would say that singular terms refer to objects and that predicate clauses refer to or express properties of those objects.

Let us for now take set theory at face value, that is, accept platonism. A theory about sets telling us which sets there are and what properties they have could take different forms, but the standardly accepted way of theorizing about sets is through an axiomatic theory. I will follow this, and when I use "set theory" in the following I am referring to ZFC, that is, Zermelo-Frankel set theory with AC. When considering the axioms of this theory, the problem is this: if pure sets are non-causal and not located in space and time, how could we know anything about them? How could we know which axioms correctly capture what sets there are? This is puzzling, and indeed it is one version of perhaps the main objection to platonism about mathematics: the epistemological challenge. Paul Benacerraf articulated this challenge very clearly in his 1973 paper. We seem to find ourselves in a tension between on the one hand taking mathematical language at face value on the grounds of simplicity and also in order to extend our usual semantic analysis to mathematical language, and on the other being committed to a nominalist (meaning wholly non-abstract) ontology in order to preserve a causally based epistemology. Allies of the latter position often want to hold onto the truth of sentences that mathematicians classically hold as true. Since they cannot take such statements at face value they have to offer some re-interpretation of those sentences that somehow only make reference to concrete entities and at the same time make the truth-values come out right.

The relation between platonism and the defense of set theoretical axioms is my main concern in this paper. Since some axioms of set theory will be important later in this paper, I will state them here and assume familiarity later:

> **Extensionality:** $\forall A \forall B[\forall x(x \in A \iff x \in B) \Rightarrow A = B]$. In words: If all and only the members of some set A are members of some set B, then A and B are identical.
> **The Axiom of the Empty Set:** $\exists B \forall x(x \notin B)$. In words: There is a set which has no members.
> **The Axiom of Choice:** ($\forall$ relation R)($\exists$ function F) ($F \subseteq R \land \text{dom } F = \text{dom } R$). In words: For every relation R, there exists a function which shares the same domain as R. (Enderton 1977:271–272)

In the literature exploring how axioms of set theory can be defended there is a distinction between what is called *intrinsic* and *extrinsic* justification (Maddy 1988:482). The first one can be said to involve mathematical intuition (Holster 2016). Or, to elaborate, one is arguing that certain claims follow from the intuitive grasp of the subject matter one is trying to describe through axiomatization. In the case of set theory this would involve the conception of what sets are like and how a theory of them should behave (Maddy 1988:482). What is usually called the *iterative conception* of set is often drawn on here. On this conception, sets are ordered in a hierarchy of stages, formed by iterations of certain set-theoretic operations, starting with the empty set. George Boolos approaches the justification problem through this route, as we will see later (Boolos 1971). His project is one of two stages. First he makes rigorous our intuitions of the iterative conception of sets. He then deduces from this the axioms of set theory. The intrinsic way of defending axioms seems, at least to me, to accord very well with a platonistic ontology. How could one even argue that there is an intuitive conception of something, or that something is intuitively true of it, if that something is not real?

But this is not perfectly clear; perhaps there can be an intuitive *conception* of something that is not real, even though this does not mean that something is intuitively *true* of it. What I have in mind here is a sense in which one perhaps could take *fictionalism* to be compatible with intrinsic motivation – the idea being that something can "intuitively fit with the fiction" that one indulges in when doing set theory. In contrast, it is clearer that a formalist position, for instance, fails to make room for intrinsic justification.

Now, there is something to this contrast, but I think that also the fictionalist approach in some sense fails to truly capture intrinsic motivation. The reason is this: it is true that *when given* a fiction, something is intuitively true *of it*. This holds also for formalism, when given a formal system, some consequences are more intuitive than others. When we are concerned with an axiom *choice* however, we are concerned with which fiction to indulge in, and correspondingly which formal system to adopt. There is no way one can make sense of the fact that some *axioms* are true on a formalist position. Only given the axioms can the formalist start playing her games more or less intuitively. Similarly for the fictionalist, there seems to be no way of making sense of privileging one fiction over others purely due to intrinsic considerations. Of course, the fictionalist may still privilege one fiction over another on the basis of *extrinsic* reasons: it makes sense to pursue and develop a fiction that is *useful*. This is analogue to the way a formalist motivates her own privileging of a given formal system.

If the above reasoning is sound, purely intrinsic justification of axioms is incompatible with any anti-realism about mathematical subject matter. I think, however, that this conclusion perhaps is too strong. When pursuing intrinsic reasoning, one typically relies on a conception of the relevant subject matter, for instance the iterative conception in the case of set theory. The fictionalist can now say that given the outlines of this conception, some axioms intuitively fit with this conception and some do not. In this sense perhaps, axioms could be motivated simply by intrinsic reasoning. I think this objection is good, but that there still is something missing. The reason for this is that it only seems to push the question one step back: how to motivate the decision to develop and axiomatize this conception rather than some other? The platonist will plausibly say that the iterative conception is *true* of something, and hence on this view the intrinsic motivation exists also at this level. The fictionalist, on the other hand, cannot say this. After all, the conception is just a fiction on this view.

Now, I do not want to push further with this issue. I think it is clear that anti-realism, in the fictionalist case for instance, can motivate some axioms intrinsically *from the iterative conception*. It is not clear, however, whether this actually is intrinsic motivation, if the reason for developing and axiomatizing that conception over others is due to considerations that have to do with usefulness. I will leave this now in order to see more closely how the intrinsic motivation works in practice.

Boolos develops the iterative conception of set in order to show how some axioms follow directly from this. This conception involves what is called stages, at which the sets are "formed" by set theoretic operations. The conception says roughly something like this: At every stage, one forms all the sets that it is possible to form. Boolos lays down axioms that purport to describe this intuitive notion of forming sets in an iterative way. From them he deduces most of the axioms of ZFC. This is an example of how he deduces the Axiom of the Empty Set:

> As there is an earliest stage, stage 0, before which no sets are formed, there is a set that contains no members. Note that, by axiom (IX) of the stage theory, any set with no members is formed at stage 0; for if it were formed later, it would have to have a member (that was formed at or after stage 0). (Boolos 1971:224-225)

This is an almost trivial example, and it should be noted that some of the axioms of ZFC have traditionally been taken to demand much less justification than others. The most striking example here is the Axiom of Extensionality, as noted by both Maddy and Boolos (Maddy 1988:483). It is presented without justification in Zermelo's first axiomatic set theory (Zermelo 1908). Boolos states furthermore that if the distinction between synthetic and analytic statements can be made sense of, this is an example of the latter (Boolos 1971:229–230).

It is perhaps natural to think that intrinsic motivation is characteristic of most foundational problems in mathematics and that this discipline enjoys some special status by building its theories on mostly intrinsically justified axioms. I.e., they are all taken to be obviously true, in contrast to the extrinsically justified hypotheses of natural science. This would be to hold some form of *exceptionalism* about mathematics, i.e. that mathematics has some very special status compared to other sciences, a status that manifests itself in a very special method and access to mathematical facts. But here Maddy is warning us:

> [...] the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters. In such cases, we find the methodology has more in common with the natural scientist's hypotheses formation and testing than the caricature of the mathematician writing down a few obvious truths and proceeding to draw logical consequences. (Maddy 1988:481)

According to her, most axioms of ZFC are justified by a mix of intrinsic, and what she calls extrinsic support (Maddy 1988:482–483). I take extrinsic justification to be all those considerations one usually builds scientific theories on, other than intrinsic support, which includes usefulness, fruitfulness, predictive power, simplicity, unity, parsimony and perhaps more. I think this is the standard way of understanding the distinction (one finds a similar view in Maddy 2011:47). Justifying something by abductive reasoning[1] is a form of extrinsic justification.

The picture we get from Maddy is one where foundational issues in mathematics are not that far away from corresponding issues in natural science. Kurt Gödel did to some extent hold a similar view, namely that the evidence for the existence of mathematical objects was analogous to the evidence we have of physical bodies:

> It seems to me that the assumption of such [abstract objects] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions [...] (Gödel 1944:456–457)

Gödel is here talking about the ontological or metaphysical aspect of mathematical theorizing, Maddy on the other hand about choice of axioms. Both think extrinsic justification plays an important part in settling which theory to adopt, and it is here that the platonists have the strongest response to Benacerraf. In taking the axioms of set theory to be justified not solely by intrinsic, but also by a range of extrinsic considerations, theorizing about set theory (and mathematics more generally) is similar in important respects to the natural sciences.

I think this way of thinking about mathematics is correct, and that the relationship between ontology and axiom choice is also one where extrinsic justifications come into play. They are heavily intertwined aspects of the same story. What I mean by this more concretely is that whichever theory one adopts has implications for the ontology one should accept, and whichever ontology one adopts will have implications for which theory to accept.

I will demonstrate this interrelatedness of mathematical ontology and axiom choice with an argument. Firstly, this argument shows how platonism could be extrinsically motivated by usefulness, and secondly how this ontology in turn might motivate the Axiom of Choice specifically based on extrinsic reasoning about ontological parsimony. All in all, the argument follows Maddy and Gödel in relying on their anti-exceptionalism about mathematics.

The Argument

1. There is a branch of mathematics that depends on Choice-principles and that is indispensable for natural science.
2. Every branch of mathematics that is indispensable for natural science has referring terms.
3. Intermediate conclusion: There is a branch of mathematics that depends on choice-principles and has referring terms.
4. Every branch of mathematics that has referring terms has terms referring to sets.[2]
5. **Intermediate conclusion:** There is a branch of mathematics that depends on choice-principles and that has terms referring to sets.
6. That the Axiom of Choice is a true axiom of set theory would be the best theory that accommodates the fact that there are branches of mathematics that depend on choice-principles and refer to sets.
7. We should adopt the best theory among those theories that accommodates the fact that there are branches of mathematics that depend on choice-principles and refer to sets.

Conclusion: Hence, we should adopt the Axiom of Choice as an axiom of set theory.

The first part of this argument draws heavily on the well-known indispensability argument. Summarized in one sentence, the argument says that mathematical statements about numbers, functions, sets and so on are indispensable for the natural sciences and that therefore we should think that mathematical statements really are about numbers, functions, sets and so on. This argument is associated first and foremost with W.V.O Quine and Hilary Putnam.

Quine was a thoroughgoing empiricist, which can seem to be in conflict with platonism for the reasons that Benacerraf stressed. Nevertheless, Quine's idea of confirmation holism leads him to think that most of mathematics is about abstract objects. Confirmation holism is the view that sensory experience cannot confirm or disconfirm a scientific hypothesis in isolation. Rather "statements about the external world face the tribunal of experience [...] only as a corporate body" (Quine 1951:41). Confirmation holism, in the context of a few other premises, leads to meaning holism, i.e. that no statement has meaning considered in isolation, but only in relation to other parts of the whole language. How might this lead to mathematical platonism? The idea is that hypotheses in science are formulated using mathematics, and hence that there are myriads of links between the hypotheses of physics, chemistry, biology and so on, and the part of the belief web that consists of mathematical statements. In this way, statements about mathematical entities are indirectly "confirmed" by experience in much the same way the existence of other theoretical entities can be confirmed (Quine 1951:44–45). This is a clear anti-exceptionalist position analogous to the quote by Gödel above.[3] We might bring the argument down to two premises as follows:

1. Classical mathematics is indispensable for natural science.
2. Classical mathematics would not be indispensable for natural science if mathematical terms did not refer.
3. Hence, mathematical terms do refer.

One might think that "indispensable" only means that if we got rid of mathematical formulations today we would throw out a lot of scientific theories as well, but that does not exclude the possibility of getting the same theories working after giving a nominalistic interpretation eliminating the platonistically committed language. That would be a non-substantial reading of the argument. The stronger reading, and the one I think is put forward by Quine and Putnam, takes "indispensable" to mean that it is principally impossible to get a nominalistic reinterpretation to work for those branches of mathematics that are indispensable for natural science, especially for real numbers and functions on them. This is the more substantial premise and reading of the argument, and the one that I will work with now. I do not have any additional reasons supporting this, other than those that have been offered before. Hence, if the indispensability argument can be refuted by giving a working nominalistic reinterpretation, then my argument would go the same way.

The second part of the argument is concerned with ontological considerations. The guiding idea behind it is that set theory has proved to be an immensely rich theory in the sense that almost all of mathematics can be *interpreted* within this theory. This leads to the idea that other branches of mathematics also could be *reduced* to set theory.

What does "interpretation" and "reduction" mean here? The former is a matter of technical results; the latter is a philosophical and ontological matter. By interpretation I mean more precisely modeling some branch of mathematics within set theory. Take basic arithmetic as an example. For a set theoretic model of the natural number structure, we might identify every natural number with the set of all smaller numbers, e.g. zero is interpreted as the empty set, one is interpreted as the set of the empty set, two as the set containing the empty set and the set of the empty set and so on, in order to get a translational scheme from one theory to another. This is the definition offered originally by von Neumann, and currently the standard way of doing it. We then proceed by defining recursion and functions on the natural numbers. From this we can prove arithmetical theorems directly in set theory. One might continue extending the translation from other branches of mathematics to set theory by defining integers as equivalence classes on pairs of natural numbers, define rational numbers as pairs of integers, use Dedekind cuts to construct the real numbers and so on. Set theory provides resources to model (almost[4]) all of mathematics. All the same, it is important to distinguish the activity of interpreting different structures within set theory from the philosophical concern of whether natural numbers, rational numbers, real numbers and the rest can be reduced to sets. By reduction I here mean literally identifying the number zero with the empty set, and not just for the sake of modeling.

Here, I have distinguished between two relations set theory bears to other parts of mathematics. The first is that it offers a place to model mathematical structures without any further commitments, and the second is that it offers an ontology that opens up the possibility of considering reduction of other mathematical entities to sets. Both of these aspects are connected to the status that set theory has as a *foundational branch* of mathematics, as mentioned in the introduction to this paper – one much more substantial than the other. It is not always clear which of these aspects philosophers and mathematicians have in mind when they reflect on this. Take for example the opening passage of Ernst Zermelo's first axiomatic set theory:

> Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions "number", "order", and "function", taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis; thus it constitutes an indispensable component of the science of mathematics. (Zermelo 1908:200)

I think this is a hint towards an intermediate view on the relation between set theory and other branches of mathematics: a view which holds that interpretations of number theory, analysis and so on within set theory are more than simply technical results, but less than reduction in the strong sense developed above. The idea is perhaps that somehow being able to model a structure within set theory gives confirmation of the existence of this structure and offers a unified way of understanding the relations between different structures. I think it is something like this that Maddy has in mind when she says the following:

> What emerged gradually – in the theory of functions, in Dedekind's constructions of the reals, in the foundations of arithmetic and elsewhere – is that set theory provides a natural arena in which to interpret the myriad structural descriptions of mathematics, to settle which are and aren't coherent. (Maddy 2011:32)

Premise 3 of my argument, however, goes beyond this view of the relation between set theory and other branches of mathematics and voices a strong reductionist view. The motivation for reductionism I take to be classical theoretical virtues: parsimony in ontology, simplicity and unity. If we can limit that part of our ontology that consists of abstract objects to sets alone, then this would be a more parsimonious ontology than having natural numbers, real numbers and so on in addition. It also offers a unified way of understanding mathematical structures and their relations; if set theoretic reductionism is true, then they are all most fundamentally studied within the framework of set theory.[5]

To end the explanation and justification of the argument, some attention should be directed at what I call choice-principles, and premise 7 and 8. It is crucial for the argument that there is some part of mathematics that is indispensable for natural science**,** and which depends on choice principles. This in turn justifies accepting choice principles in set theory, since without it we would not get a reduction of those branches of mathematics to set theory. I do not have very much to bring forward in defense of the claim that the mathematics that is indispensable for natural science depends on choice principles; I lean mostly on this short line from Maddy on the usefulness of Choice:

> Since then [1908 and Zermelo's first papers on Choice and the Well-ordering principle] it has become clear that the Axiom of Choice and its equivalents are essential not only to set theory but to analysis, topology, abstract algebra and mathematical logic as well. (Maddy 1988:488)

Analysis and topology are very important fields of mathematics where application in science is concerned, and this is what my defense of premise 1 mainly rests on. One might wonder why I go the route through premise 7 and 8 in order to argue for AC, why appeal to the best explanation? The reason for this is that one could probably adopt weaker versions of the Axiom of Choice, i.e. implying the existence of a choice set for sets with sufficiently small cardinals, and still get a reduction of those branches of mathematics indispensable for science. The Axiom of Choice, however, is not limited in this way by the size of sets, and the reason for adopting it is theoretical virtues. Adopting a "full fledged" AC gives a simpler and more elegant theory. One could, for instance, say that putting a limit on the choice axiom would be an unmotivated and arbitrary theoretical limitation. It is also worth noting that the quote by Maddy suggests another way of understanding the indispensability of mathematics: one could simply take AC to be indispensable for mathematics in its own right, without taking application in science into consideration. In this way one could say that since AC is essential in so many mathematical fields, this in itself justifies taking it as a true axiom of set theory. I think this strategy is sound as well, but I think there is something interesting still in pursuing the indispensability argument that targets application in science as I do here.

This concludes my explanation and defense of the various premises of the main argument. To summarize, the argument holds that that mathematics depending on choice principles are indispensable for science (and mathematics in itself), and that mathematics that is so indispensable should have referring terms or "be about something." This gives us a platonist reading of mathematics and set theory. Premise 4 says in effect that if we have platonism, then we should have a reduction, in the strong sense, of all of mathematics to set theory. The reason for this was especially parsimony in ontology, as already mentioned. This in turn justifies accepting a sufficiently strong choice axiom in our set theory. Accepting the strongest, the Axiom of Choice, gives the simplest and most elegant theory.

All this serves to show that one can defend axioms of set theory in a whole range of ways while assuming platonism. Some axioms, like Extensionality, have traditionally been defended intrinsically, something that concurs very well with platonistic ontology. Further, there is room for arguing abductively about what axioms to defend given platonism. I will now discuss some anticipated problems with my main argument.

Possible Objections

Platonism and the Indispensability argument are associated with taking mathematical language at face value. With a set theoretical reductionist view, however, one might think that this is no longer the case and that this is problematic. Granted, one will need a reinterpretation in some sense, but the way I see it this is not a problem. The set theoretic interpretation is not threatened by worries of failure, like the nominalistic program might be. That it works out is a purely technical point that has been studied for a long time. One might further worry that taking numerals, for instance, to refer to sets make for a counter-intuitive reinterpretation of classical mathematics. If we had strong intuitions to this effect, this would be a legitimate concern – but do we really have strong intuitions for thinking that the referent of for instance "3" could not be a particular set? Could we not tell the story more like those of other scientific discoveries? It could go like this: We knew for a time that "3" referred to an object, and now that we know more about that object, it is a set. I think this looks promising to begin with, but an important challenge to it has been made.

This challenge is one that Benacerraf influentially discussed in his 1965 paper. In effect, the argument turns on the distinction I made earlier between *interpreting* mathematical objects in set theory and *reducing* mathematical objects to set theory. The claim is that reduction is meaningless since there are so many equivalent (in all relevant respects) interpretations. We could identify the natural numbers with the set of all smaller numbers, following von Neumann. Or we could identify zero with the empty set, one with the set of the empty set, two with the set of the set of the empty set and so on.[6] Both of these, and infinitely many others, would serve our purpose equally well. Benacerraf then claims that there is no way, at least no rational or non-arbitrary way, of choosing one over the other as the referents of our numerals, and from this he argues that it is not the case that one is the correct one. This goes against the set theoretic reductionist project in general, and hence of premise 4 in my argument. Benacerraf gives two reasons for why there can be no rational choice between the two (and other) accounts: one meta-semantic and one epistemic. In Benacerraf's own words the first goes as follows:

> There is no way connected with the reference of number words that will allow us to choose among them, for the accounts differ at places where there is no connection whatever between features of the accounts and our uses of the words in question. (Benacerraf 1965:285)

The epistemic argument follows the idea that if 3 was really some particular set, then which one it is would be impossible for us to know. That there is such a fact, but that it is unknowable would be "hardly tenable" (Benacerraf 1965:284). Let us for now call Benacerraf's challenge for the "identification problem".

The kind of set theoretic reductionism that is put to use in this paper is not very popular today, and this for a range of different reasons. For instance, it is a live question whether category theory in some sense offers a better foundation for mathematics, and also for set theory. Arguably, however, people did for a long time talk in a way that would suggest that numerals, functions, etc. literally referred to sets (cf. the quote by Zermelo above). The identification problem challenges this kind of talk taken straight on: what happens if we take the set theoreticians talk of numbers as sets literally? Moreover, the identification problem has been very influential – consider this paragraph from Michael Potter:

> Another theme has recurred throughout the last century: the fact that the theory of real numbers, and by extension most of the rest of mathematics, can be interpreted in set theory has been taken to show that they can be thought of in some significant sense as being *part* of set theory. The reasons that have led people to think this can be grouped into two sorts, one (in my view) distinctly more promising than the other.
>
> The less promising argument takes as its ground a principle of ontological economy (Occam's razor): since the specification for the natural numbers which we drew up in the first stage of the process listed enough of their properties to characterize them up to isomorphism, and since we have shown that there is a *set* which has just these properties, it would multiply entities beyond necessity to suppose that the natural numbers are anything other than the members of the set-theoretic model. That this is a bad argument irrespective of the general merits of Occam's razor was lucidly exposed by Benacerraf (1965) who pointed out that the non-uniqueness of the set-theoretic model fatally flaws the claim that its members are really the natural numbers: if no one pure set has a privileged claim to that title, then none can have title at all. (Potter 2004:150)

The other approach that Potter has in mind is the way in which set theory provides a unified way of understanding and relating different mathematical structures.

Nevertheless, the point I want to make by highlighting the quote by Potter is that Benacerraf's objection is important. There might be other (even stronger) reasons for thinking that other branches of mathematics do not reduce to set theory, but here I will put those aside: I assume that the motivation for reductionism that comes from simplicity and parsimony is legitimate, and also that Benacerraf's challenge is the most important objection to it. I will therefore focus my efforts in the last part of this paper on the discussion of relevant counter objections to the identification problem, and not consider challenges that could be made to reductionism generally.

I think there are two possible routes for countering the identification problem, and I will briefly explore both of them here. The first involves rejecting Benacerraf's argument, while the second is to grant that Benacerraf is right in holding that these kinds of reductions do not make sense, but that there are other ways of identifying the referent mathematical terms that is not a random or irrational choice among equally good options in the same sense as those above.

I present the first route first. The epistemic argument seems to rely on the following principle: all mathematical facts should be in principle knowable. Is this plausible? It certainly is not in most areas of inquiry besides mathematics. Take for example the case of the orbiting time of planets. If we consider planets far beyond our observational reach, then their orbiting time would be unknowable to us. It seems strange to say in this case that if something is unknowable, then there is no fact of the matter.

Perhaps the principle should be restricted to concern only facts about reference, i.e. that all facts concerning the reference of mathematical terms should in principle be knowable. This could be motivated by the thought that somehow reference is special because it is constituted by linguistic practice and hence in some way is constituted by us. But if one holds externalist positions about reference, we see that this reason also could be challenged. We do not need to know what we refer to in order to refer to them. Other factors might still decide on one determinate reference. This also translates into an objection to the meta-semantic argument. There might be nothing in the layman's use of numerals that may single out a determinate reference, but this does not exclude the possibility that other factors, or the use of some privileged group of people, do just that. Perhaps some expert group in our community thinks one interpretation is much more practical or reasonable to use (something that actually seems to be the case, at least for natural numbers, given how standard the von Neumann interpretation has become). This expert group could with this establish a robust reference relation for our community. If this were so, then both the meta-semantic and the epistemic argument would fail. Overall, both arguments can be criticized on the ground that they seem to invoke some special access to mathematical facts, where both reference and knowability are concerned.

What if we do not go down this route, but rather find Benacerraf's arguments quite plausible? Is there no way of upholding premise 4, even though the reductionist project – as he envisages it – fails? I do not think that there is, unless we change premise 4 to read like this: Every branch of mathematics that has referring terms has terms referring to sets or *classes*.

The reason for this is as follows. If we want to identify the referent of numerals and other mathematical terms with particular sets and avoid the accusation of making random choices among equivalent interpretations, we could identify the natural numbers, for instance, not with any particular set theoretic instance, but rather with the set of every such structure isomorphic with the natural numbers. If we can develop an isomorphism relation that take every natural number structure in set theory to all other natural number structures in set theory, then the set of things such related will be a candidate for a non-arbitrary selection for the natural number structure within set theory, or so I claim at least. For individual numbers we could do something similar, namely, to have an isomorphism relation relate everything that fills the 2-role in one natural number structure to all other things filling equivalent roles in isomorphic structures. The referent of *2* could then be the set of those things such related.

The problem with this approach, however, is that there are no such relations, and hence no such sets. The reason for this is that such sets would be too big, i.e. from them we could obtain the universal set. However, they could perhaps be proper classes.[7] The second option I pointed to above is then to identify natural numbers, real numbers and so on with the proper class containing the set that plays the relevant role in every set theoretic interpretation of the structure that object belongs to.

The cost of going this route, if it is at all possible, is that it makes the reductionist project less parsimonious than we might have hoped. Instead of making do with only one category of abstract mathematical objects (sets), we now have two (sets and classes). This should not worry us too much; we still seem to get a lot of theory for a very cheap ontology. This approach, together with the possibility of rejecting Benacerraf's arguments, should leave premise 4 with some hope.

Conclusion

In this paper I have looked at how the defense of axioms of set theory looks in light of the epistemic challenge to Platonism from Benacerraf. How can we know which axioms to adopt if we think they purport to describe abstract objects not causally related to us in any way?

I started this investigation by drawing on the distinction borrowed from Maddy between intrinsic and extrinsic motivation. I pointed out that intrinsically motivated axioms combine well with Platonism, but that they do not shed much light on how we can have knowledge of abstract objects. I then turned to extrinsic motivation. The main part of the paper was to present an argument that demonstrated how one could hold Platonism and motivate the Axiom of Choice specifically through abductive considerations.

This supports the position I called anti-exceptionalism, a position that Maddy and Gödel both endorsed (though different versions). According to this position, foundational issues in mathematics are analogous to those of other sciences, drawing on a range of extrinsic support for their axioms in addition to the intrinsic justification one might have.

NOTES

[1] I take abduction here to be synonymous with inference to best explanation or theory.

[2] We will see later that there might be reasons for rewriting this premise (and the subsequent premises accordingly) to read: Every branch of mathematics that has referring terms has terms referring to sets or classes.

[3] There are of course many important differences between Quine and Gödel (and Maddy also, for that sake). The specific way in which I take them to be comparable here is the sense in which they think that the justification we have for the existence of mathematical entities is similar to that we have for theoretical entities in general.

[4] One commonly notes that category theory is a possible exception, but I will not examine whether this is in fact so, or to what extent category theory poses a challenge to the status that set theory has as the "foundation of mathematics".

[5] This does not mean that it is obligatory to do all of mathematics within set theory if reductionism holds. One could and should still use the more practical frameworks that one uses today. The point is simply that if all of mathematics is really about sets, then set theory offers the most fundamental description. This is similar to the way in which other natural sciences might be reducible to fundamental physics; if so is the case, this does not mean that one should stop doing chemistry or biology, it only means that most fundamentally what it is all about is that which fundamental physics describes.

[6] Which is actually what Ernst Zermelo once proposed.

[7] The attractiveness of this approach relies of course heavily on the availability of a class theory that can do the relevant work. I do not actually know to what extent this is a simple task, but I will leave that aside for now.

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