FRA FORSKNINGSFRONTEN

HOW FINE-GRAINED IS REALITY?

By Peter Fritz

1. Barbers and Sets

Here is a well-known puzzle: Say there is a village with a barber. Some (male) villagers shave themselves; others are shaved by the barber. In fact, the barber shaves all and only those who don't shave themselves. Who shaves the barber?

This is of course a trick question: The answer is that there cannot be such a village. If every villager is such that the barber shaves him if and only if he does not shave himself, then this must also hold for the barber: the barber must shave himself if and only if he does not shave himself. But that is a contradiction.

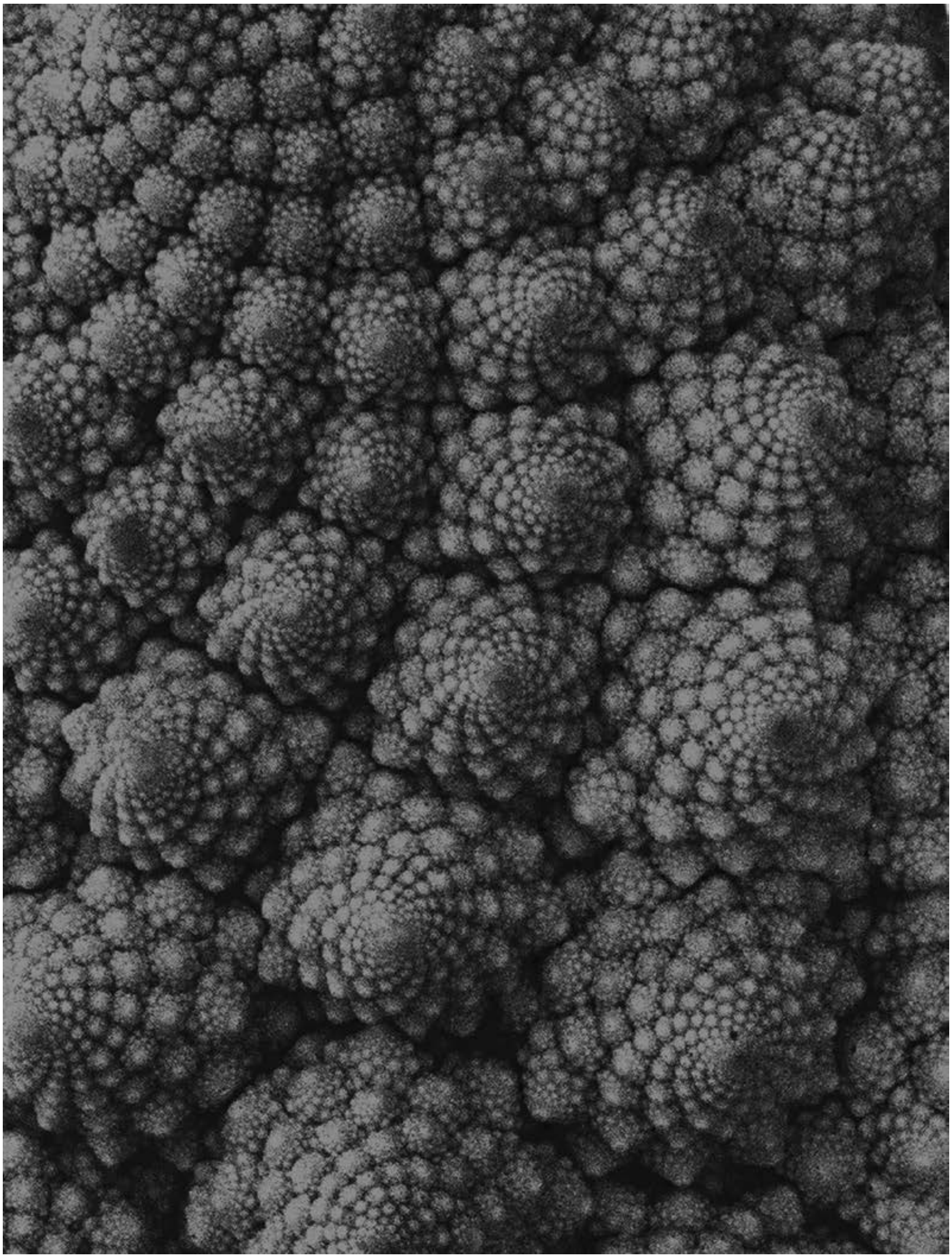
A similar puzzle arises from the notion of a set, the standard notion of a collection in mathematics. Consider the set of sets which are not members of themselves. Call it the Russell set, after Bertrand Russell, who came up with the argument.[1] Unpacking the description, this is the set *R* such that for all sets *x, x* is a member of *R* just in case *x* is not a member of *x*. If this holds for all sets, then it must hold for *R*. So *R* is a member of *R* just in case *R* is not a member of *R*. But that is again a contradiction.

Standard set theories in mathematics respond to Russell's argument just as we did to the barber puzzle: They conclude that there cannot be such a thing as the set of sets which are not members of themselves. Various stories have been told to motivate this. A popular one holds that sets are constructed in a never-ending series of stages: At first, there are no sets. To this, we add all the sets whose members are only non-sets. Then we add the sets whose members are the non-sets and the sets added at the first stage. And so on: at each stage, we add the sets whose members are non-sets or sets constructed previously. No set contains itself, but we never form the set of all sets not containing themselves, because we never form the set of all sets: there is no last stage – we always continue adding more sets. While the stages and the process of adding may be somewhat mysterious, the resulting theory of sets turns out to work well for many mathematical purposes.[2]

2. Properties and Propositions

Russell's argument can also be formulated for properties: consider the property of being a property which does not have itself (as one of its properties). Call this the Russell property. As before, this leads to a contradiction if we consider whether the Russell property has itself: since the Russell property is the property of being a property which does not have itself, the Russell property has the Russell property just in case it is a property which does not have itself. Another contradiction. And as with our barber and the Russell set, we could conclude that there is therefore no such thing. But that does not sit well with some of the uses we want to make of properties in philosophy.

One such use is in the philosophy of language. Names, like "Socrates", refer to things: "Socrates" refers to Socrates. Predicates, like "is snub-nosed", express properties: "is



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snub-nosed" expresses the property of being snub-nosed. Sentences express propositions: "Socrates is snub-nosed" expresses the proposition that Socrates is snub-nosed. And we can conclude this last observation from the earlier two: If a predicate expresses a property and a name refers to a thing, then the sentence formed by applying the predicate to the name expresses the proposition that the thing has the property. In general, whatever phrase we compose from words in grammatical ways, we can compose what these words express in analogous ways to obtain what the phrase expresses.

That, at least, are rudiments of a plausible theory of how languages like English work. But they lead to trouble with the Russell property: "being a property which does not have itself' certainly seems to be a phrase composed grammatically from meaningful words. How, then, could it fail to express a property, as it must if we want to hold that there is no Russell property?

Many theories in linguistics and philosophy therefore respond to Russell's argument for properties in a different way. That way sets aside, to a degree, our intuitive notion of a property, and looks at the role this notion plays in theorizing about language. Consider "Possibly, Socrates is snub-nosed". This attributes possibility to the proposition that Socrates is snub-nosed. Possibility therefore seems to be some kind of property. But "Possibly, Socrates" is ungrammatical. This string of words tries to attribute possibility to Socrates, but fails to express a proposition. A reason is not hard to find: while it makes sense to think of possibility as a property of propositions, it makes no sense to think of it as a property of individuals. Consider now the expression "is snub-nosed is snub-nosed". This is again ungrammatical. It tries to attribute a property of individuals to a property, and fails.

A picture emerges: The entities which we express with our words divide into categories which correspond to the grammatical categories of these words, and the ways in which we can combine these entities correspond to the ways in which we can combine these words. Just as a predicate can only compose with a name to form a sentence, the property it expresses can only be had by a thing. And we must distinguish between different kinds of properties: predicates like "snub-nosed", which compose with names to form sentences, express properties of individuals, but words like "possibly", which compose with sentences to form sentences, express properties of propositions.

The problem with the property of being a property which does not have itself is thus a kind of category confusion. For example, consider properties of things, like being snub-nosed. Just as "is snub-nosed is snub-nosed" is ungrammatical, it makes no sense to ask whether being snub-nosed is snub-nosed: not because properties don't have noses, but because a property of things is not the kind of property that a property of things could have – properties of properties of things are in a different category than properties of things. The fact that the English "a property which does not have itself" is still grammatical is thus a sort of defect of English. Logicians and linguists have therefore worked out artificial languages in which words are assigned strict categories, governing the ways in which they can be combined. The categories are often called "types", and the resulting systems "type theories" (Frege 1879).[3]

Type theories are useful in the systematic study of language. But they are also useful in metaphysics: if things, propositions, and different kinds of properties separate into a rigid hierarchy of "types", then these type-distinctions are well-worth making when theorizing about what things, propositions and properties are like in very general respects, as we do in metaphysics.

3. Structure

What are things, propositions and properties like, in very general respects? Consider propositions. One way of sharpening the question is to ask what it takes for two sentences to express the same proposition. Is the proposition that Socrates is snub-nosed the proposition that Aristotle is sitting? We're inclined to think not. Whenever we have two simple predications – sentences composed of a single predicate and a single name – we're inclined to think that the two sentences can only express the same proposition if their predicates express the same property and their names refer to the same things. In general, it seems plausible that two sentences can only express the same proposition if they have the same grammatical structure, and any word of the first sentence expresses the same as the word in the corresponding position in the second sentence. Call this the "structured proposition view", since it holds that propositions inherit the structure of the sentences expressing them. Surprisingly, we can show that this view is inconsistent![4]

The argument is in some ways similar to Russell's argument against the existence of the set of non-self-membered sets. In fact, it was also discovered by Russell (1903, Appendix B), to be rediscovered later by John Myhill (1958), which is why it is sometimes called the 'Russell-Myhill argument'.[5]

Take an arbitrary proposition, say the proposition that

2 + 2 = 4. Consider the following property of propositions:

> To be a *stone-thrower* is to be a proposition which attributes to 2 + 2 = 4 some property which it itself lacks.[6]

So a proposition *p* is a stone-thrower just in case there is some property of propositions such that *p* doesn't have it, and *p* is the proposition that 2 + 2 = 4 has it. For example, the proposition that 2 + 2 = 4 is distinct from itself is a stone-thrower, since there is a property of propositions – namely being a self-distinct proposition – which it attributes to 2 + 2 = 4 but doesn't have itself.

Now let the Russell proposition be the proposition which says that 2 + 2 = 4 is a stone-thrower. We will show that there is some property of propositions distinct from being a stone thrower, such that the proposition that 2 + 2 = 4 has that property is the same as the proposition that 2 + 2 = 4 is a stone-thrower. Since the two properties are distinct but the resulting propositions are the same, the structured proposition view must be false. This is the argument:

> We first argue that the Russell proposition is a stone-thrower, by deriving a contradiction from the assumption that it isn't. So assume that the Russell proposition is no stone-thrower. Then there may not be any property which it attributes to 2 + 2 = 4 but lacks itself. But since it attributes to 2 + 2 = 4 being a stone-thrower, it may not itself lack the property of being a stone-thrower. Therefore it must be a stone-thrower, contradicting our assumption that it isn't.
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> So the Russell proposition is a stone-thrower. Thus it attributes to 2 + 2 = 4 some property *F* which it itself lacks. But since the Russell proposition has the property of being a stone-thrower, *F* must be distinct from being a stone-thrower. Thus the Russell proposition, the proposition that 2 + 2 = 4 is a stone-thrower, is also the proposition that 2 + 2 = 4 is *F*, for some property *F* distinct from being a stone-thrower. And that is what we needed to show.

Note that all the properties appealed to in this argument are properties of propositions, and are only ever attributed to propositions. The distinctions between different kinds of properties which we used to argue that there is no such thing as the Russell property – described as the property of properties which don't have themselves – cannot be used to argue that there is no such thing as the Russell proposition. The argument also doesn't rely on any ambiguity of English: we can carry out the argument in formal type theories used by linguists and philosophers which are demonstrably consistent.

What does this mean? One may take it to show that something went wrong with the story we developed above about the entities expressed by phrases and the different types into which they fall. Maybe we went astray when we derived these types from English grammar but then criticized English for not distinguishing between them rigorously enough. Maybe. But maybe not. Maybe the Russell-Myhill argument just shows something interesting and surprising. That a purely formal argument would establish a surprising conclusion is no reason to think that something has gone wrong: You might have thought that all infinite sets are of equal size. They are not.[7] You might have thought that you can write a computer program which outputs all and only the true arithmetic sentences in predicate logic. You can't.[8] You might have thought that a geometrical ball cannot be separated into a finite number of segments which can be reassembled into two balls of the same radius as the original. It can be done.[9] And so on.

4. Necessity

So, what if propositions are not structured? A general way of thinking about the relation between sentences and propositions is in terms of how many of the features of sentences are reflected in the propositions they express. If a feature of sentences is reflected in propositions, then two sentences differing in this feature will express different propositions; if not, they may express the same proposition. Thus, the more features of sentences are reflected in propositions, the more finely propositions are individuated, or, as it is sometimes put, the more fine-grained propositions are. The view that propositions are structured in the sense of reflecting the structure of the sentences expressing them is therefore a view attributing to propositions a high degree of fineness of grain. The conclusion of the Russell-Myhill argument thus puts a bound on how finely propositions may be individuated.

Let's consider the other direction: are there plausible bounds on how *coarsely* propositions may be individuated? The most extreme view in this direction – the most coarse-grained view of propositions – is presumably the view that there are exactly two propositions: the True and the False. This is a radical view, and it might strike one as absurd: Can't I, one might reason, believe the proposition that 2 + 2 = 4 without believing the proposition that there are Higgs bosons, even though both propositions are true? But, as Gottlob Frege already observed (1892b), such attributions have to be treated with care.[10] The fact that some citizens of the USA are opposed to Obamacare but not opposed to the Affordable Care Act does not show that Obamacare is distinct from the Affordable Care Act.[11] As examples like this demonstrate, the truth of ascriptions of belief and opposition, along with may other similar intentional states, is not only a matter of the propositions and things that are said to be believed or opposed themselves, but also a matter of how we refer to them. It might thus be that the proposition that 2 + 2 = 4 is the proposition that there are Higgs bosons, even though someone believes that 2 + 2 = 4 without believing that there are Higgs bosons.

Nevertheless, many contemporary philosophers believe that propositions are more finely individuated than truth-values. Many think that there are distinct truths and distinct falsehoods. One reason for this is due to Saul Kripke's work on necessity (1980 [1972]). Some truths, such as 2 + 2 = 4, are necessary – they could not have been false. Other truths, such as the fact that I live in Oslo, are not necessary – they could have been false. Kripke can be understood as arguing that such necessity ascriptions are a matter of *what* is being represented, not *how* it is being represented. Key in this argument is his claim that on his way of using the term, "necessary" does not express anything like "can be known by pure reasoning" (sometimes called "being a priori"): I may be able to know by pure reasoning that Obamacare is Obamacare, but not that Obamacare is the Affordable Care Act; nevertheless, the proposition that Obamacare is Obamacare is plausibly the proposition that Obamacare is the Affordable Care Act. This is not so in the case of necessity: to say that Obamacare is necessarily a certain way, just is to say that the Affordable Care Act is necessarily that way.

On the basis of such considerations, it is now widely held that sentences *S* and *T* only express the same proposition if it is necessary that *S* just in case *T*. This gives us a way of arguing that there are infinitely many propositions: Take any distinct natural numbers *n* and *m*. There could have been exactly *n* donkeys, in which case there would not have been exactly *m* donkeys. So it is not the case that necessarily, there are exactly *n* donkeys just in case there are exactly *m* donkeys. Thus for any distinct natural numbers *n* and *m*, the proposition that there are exactly *n* donkeys is distinct from the proposition that there are exactly *m* donkeys. Since there are infinitely many natural numbers, there are infinitely many propositions. Just as the Russell-Myhill argument puts an upper bound on how finely propositions may be individuated, Kripke's metaphysical notion of necessity puts a lower bound on it.

5. Logical-Metaphysical Explorations

The two bounds on how finely propositions may be individuated leave much to be settled. Consider the question whether every proposition *p* is identical to its conjunction with itself. Claiming that *p* must always be the same as *p* and *p* is consistent with our lower bound on how finely propositions are individuated, since it is not possible for *p* to be the case without *p* and *p* being the case, or *vice versa*. Likewise, claiming that some *p* is distinct from *p* and *p* is consistent with our upper bound on how finely propositions are individuated, since the claim in no way implies the full view of structured propositions which the Russell-Myhill argument showed to be untenable. So: is it true or false?

There are many such questions, and similar questions arise for properties; it is far from obvious how to answer them. For those who take propositions and properties to be important constituents of reality, the study of which is the business of metaphysics, the following thus emerges as a central question of metaphysics: How finely are propositions and properties individuated? Or, as the title of this article puts it: How fine-grained is reality?[12] Although the arguments and ideas which I have used to motivate this question have been widely discussed in philosophy for close to half a century, the question has rarely been addressed head on. Many philosophers have either used type theory but shied away from attributing any serious metaphysical significance to it, or worked on the metaphysics of propositions and properties but shied away from using type theory to articulate it.

The sketch of the Russell-Myhill argument given here makes it clear that any attempt to answer the question has to be formal, and must be developed rigorously to ensure consistency. The pitfalls of attitude ascriptions in making judgements about the distinctness of propositions make it clear that we must use our best theories in the philosophy of language to avoid confusing what is being represented with what is doing the representing. And finally, the impact of Kripke's considerations on necessity makes it clear that we must appeal to distinctly metaphysical considerations to decide between competing views.

Investigating the fineness of grain of propositions and properties is thus a difficult task, which requires combining state-of-the-art research in logic, the philosophy of language and metaphysics. But it will be worth our effort: Advances on these issues offer glimpses into the fundamental structure of reality.[13] A vast landscape of views lies between the two bounds laid out here, most of which is untouched by the human mind, waiting to be explored.

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NOTES

[1] Russell discovered the argument in 1901 as a way of deriving a contradiction in a formal system developed by Gottlob Frege. He communicated it to Frege in a letter from 1902; their correspondence can be found in van Heijenoort (1967). The argument was discovered independently by Ernst Zermelo around the same time (Irvine and Deutsch 2016).

[2] This is often called the 'iterative conception of set' (Boolos 1971). It can be used to motivate different formal systems; the most widely used is Zermelo-Fraenkel set theory with the Axiom of Choice, abbreviated 'ZFC'. Enderton (1977) gives a rigorous but accessible introduction to this set theory.

[3] The basic idea of such type distinctions is already implicit in the first formulation of predicate logic, Frege (1879), and worked out more fully in in Frege (1891, 1892a). Note that these predate Russell's argument: Frege's type distinctions were based on the functions of words of different grammatical categories, rather than being introduced as a response to the argument. Why then did Russell's argument apply to Frege's system? Frege's main project was to develop a purely logical theory of arithmetic, set out in Frege (1884) and worked out in Frege (1893/1903). (Translations of Frege's works are available, e.g., Frege (1953) and Frege (1980).) As part of this theory, Frege defended a principle he called Basic Law V, which roughly says that for every property, there is a unique corresponding thing, its extension. A variation on the Russell set argument uses Basic Law V to derive a contradiction in Frege's system. Type distinctions continued to play an influential role in logic, e.g. in Whitehead and Russell (1910–1913) and Church (1940). They were central in Richard Montague's pioneering work in formal semantics (Montague 1974), and continue to be widely appealed to in linguistics (Heim and Kratzer 1998).

[4] Admittedly, I am simplifying things a little. Okay, a lot. As many have argued, we have good reason to complicate the story about language I have told, and associate an expression not with a single content (a proposition, property etc.) but several contents of different kinds. This means that it somewhat more difficult to see what to conclude from the argument described below. I discuss this in detail in Fritz (unpublished), where I argue that for the notions of propositions and properties which are central for metaphysics, the conclusions I draw here apply.

[5] The version I'm giving here is a variant taken from recent discussion, in Dorr (2016) and Goodman (forthcoming).

[6] I owe the term 'stone-thrower' to Nathan Salmon, via Cian Dorr.

[7] This was proven by Georg Cantor, published in 1891, and is often just called 'Cantor's Theorem'.

[8] Since there are infinitely many such sentences, 'outputting all' does not mean that the program has to write infinitely many sentences in a finite length of time. What is required is that for each such sentence, there is some time after which it has been produced. The result was proven by Kurt Gödel, published in 1931, and usually called his 'First Incompleteness Theorem'.

[9] This was proven by Stefan Banach and Alfred Tarski, published in 1924, and is often called the 'Banach-Tarski Paradox.'

[10] The view that there are precisely two propositions is in fact relatively naturally attributed to Frege.

[11] This example was suggested to me by Clas Weber.

[12] This way of formulating the question is taken from Goodman (forthcoming).

[13] I believe that advances on these issues will also be useful for much more down-to-earth purposes. But it would take more than a paragraph or two to explain this, and so it will have to wait for another occasion.